Complex Analysis and Related Topics

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Amsterdam, 27-29 January 1993

edited by

J.J.O.O. Wiegerinck

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Preface

On January 27 – 29, 1993 a conference was organized on the occasion of the seventieth birthday of Professor Jaap Korevaar. It was held at the University of Amsterdam under the name "New developments in complex analysis and related topics". Professor Korevaars broad interest in mathematics was reflected in the scope of the conference, ranging from wavelets to several complex variables.

The participants of the conference "decided" that the conference proceedings should aim at giving a good account of the lectures, while allowing the lecturers maximal freedom in their choice of format. Thus the proceedings should be informal and that is what they have become: One will find contributions ranging from extended abstracts to full grown papers. An advantage for myself should be mentioned. The job of editing was almost restricted to asking people for their contributions and putting them into a neat row.

The major part of the lectures is present in these proceedings. The reader will find that the papers by G. Hedstrom and A. Lukashov do not match lectures in the program. We have included these because such lectures would have been given if the circumstances would have been a little different.

The papers of P. J. Braam and J. J. Duistermaat, R. G. M. Brummelhuis, C. K. Chui and X. Shi, W. Hayman, G. Hedstrom, T. L. McCoy and A. B. J. Kuijlaars, G. G. Walter are reprinted from Indagationes Mathematicae, N.S., 4(4), 1993, an issue containing papers dedicated to Prof. Korevaar. They were included to give a more complete picture of the conference. I am grateful to the Koninklijke Nederlandse Akademie van Wetenschappen for permitting reproduction.

No conference can succeed without the cooperation and effort of many people. It is my pleasure to thank on behalf of my fellow organizers all participants for making the conference a succes and to thank the lecturers for their clear and inspiring presentation. The conference has been made possible by the financial support of the following organizations: European research office United States Army, Koninklijke Nederlandse Akademie van Wetenschappen, Office of Naval research European Office, Shell Nederland B.V., Vakgroep Wiskunde van de Universiteit van Amsterdam, Wagons-lits reizen and het Wiskundig Genootschap. Their support is appreciated very much. The Departments of Mathematics of the sister Universities: Delft, Leiden, Nijmegen and the Vrije Universiteit at Amsterdam have been helpful in many ways for which the organizers are grateful. Secretarial work was done skillfully by Philo Zijlstra, Arno Kuijlaars did a great job on the conference program. I thank them both. Finally I thank my fellow organizers Fred van der Blij and Rien Kaashoek for a pleasant and efficient cooperation.

Jan Wiegerinck

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Program

Wednesday, January 27

10:00 - 11:00	W. Rudin (Madison) *Holomorphic embeddings of* $\mathbb{C}$ *in* $\mathbb{C}^n$
11:15 - 11:55	H.J. Alexander (Chicago) *Linking and holomorphic hulls*
12:05 - 12:45	B. Berndtsson (Göteborg) *Estimates for the* $\bar{\partial}$*-equation in the one-dimensional case*
14:00 - 15:00	L. Zalcman (Ramat Gan) *Normal families revisited*
15:15 - 15:55	W.K. Hayman (York) *Integrals of analytic functions along two curves*
16:15 - 16:55	A.A. Gončar (Moscow) *Potentials and rational approximation*
17:10 - 17:30	B. Jöricke (Berlin) *Envelopes of holomorphy and CR invariant subsets of CR manifolds*

Thursday, January 28

9:00 - 10:00	W.A.J. Luxemburg (Pasadena) *Renewal sequences and the diagonal elements of the powers of positive operators*
10:15 - 10:55	J.J. Duistermaat (Utrecht) *Normal forms of symmetric systems with double characteristics*
11:05 - 11:45	P.L. Butzer (Aachen) *Bernoulli and Euler "polynomials" with complex indices and the Riemann Zeta function*
12:00 - 12:40	T.H. Koornwinder (Amsterdam UvA) *Uniform multi-parameter limit transitions in the Askey tableau*
14:00 - 15:00	N.G. de Bruijn (Eindhoven) *Convolutions of generalized functions, applied to diffraction theory of crystals and quasicrystals*
15:15 - 15:55	I. Gohberg (Tel Aviv and Amsterdam VU) *Time dependent version of results in complex analysis and operator theory*
16:15 - 16:55	Z. Ciesielski (Sopot) *Fractal functions and Schauder bases*
17:10 - 17:30	A.G. Sergeev (Moscow) *Complexifications of invariant domains of holomorphy*

Friday, January 29

9:00 - 9:50	R.G.M. Brummelhuis (Leiden) *Approximate logarithmic convexity of means of solutions of elliptic equations*
10:00 - 10:20	T.L. McCoy (East Lansing) *Answer to a query concerning the mapping* $w = z^{1/m}$
10:40 - 11:00	G.G. Walter (Milwaukee) *Analytic representations of distributions using wavelets*
11:10 - 12:10	C.K. Chui (College Station) *Wavelets and affine frame operators*
13:30 - 14:30	T. Ganelius (Stockholm) *Tauber, Korevaar and the true nature of mathematics*
15:00 - 16:00	J. Korevaar (Amsterdam UvA) *Living in a Faraday cage*

TIME-VARYING GENERALIZATIONS OF INVERTIBILITY AND FREDHOLM THEOREMS FOR TOEPLITZ OPERATORS

A. Ben-Artzi, I. Gohberg, M.A. Kaashoek

Dedicated to J. Korevaar on the occasion of his 70-th birthday,

with respect and admiration.

The three main theorems in the theory of block Toeplitz operators that deal with invertibility, Fredholm properties and index, and with factorization of the symbol, are generalized to a class of non-Toeplitz operators. The operators in this class may be described as input-output operators of time-varying linear systems. Dichotomy of difference equations plays an important role.

0. INTRODUCTION

Let $T_\Phi = [\Phi_{k-j}]_{k,j=0}^{\infty}$ be a block Toeplitz operator with $m \times m$ matrix symbol

$$\Phi(z) = \sum_{\nu=-\infty}^{\infty} z^\nu \Phi_\nu, \qquad |z| = 1. \tag{0.1}$$

The Fourier series expansion in the right hand side of (0.1) is assumed to be absolutely convergent. We consider T_Φ as an operator on ℓ_m^2, the Hilbert space of norm square summable sequences with entries in $\mathbb{C}^m$. It is well-known (see [GKr]; also [GF]) that T_Φ is Fredholm if and only if $\det \Phi(z) \neq 0$ for each z on the unit circle $\mathbb{T}$, and in this case the Fredholm index of T_Φ, $\operatorname{ind} T_\Phi$, is the negative of the winding number relative to the origin of the curve parametrized by the function $t \mapsto \det \Phi(e^{it})$. For the invertibility of T_Φ it is necessary and sufficient ([GKr]) that Φ admits a right canonical factorization relative to the unit circle, that is, Φ factorizes as

$$\Phi(z) = \Phi_-(z)\Phi_+(z), \qquad z \in \mathbb{T}, \tag{0.2}$$

where Φ_+ and $\widetilde{\Phi}_-$, $\widetilde{\Phi}_-(z) = \Phi_-(z^{-1})$, are $m \times m$ matrix functions which are analytic on the open unit disc $\mathbb{D}$, continuous on $\mathbb{D} \cup \mathbb{T}$, and their determinants do not vanish on $\mathbb{D} \cup \mathbb{T}$. Furthermore, given the factorization (0.2) we have $T_\Phi^{-1} = [\Gamma_{kj}]_{k,j=0}^\infty$, where

$$\Gamma_{kj} = \sum_{r=0}^{\min(k,j)} \gamma_{k-r}^+ \gamma_{r-j}^-,$$

with

$$\Phi_-(z)^{-1} = \sum_{j=-\infty}^{0} z^j \gamma_j^-, \quad \Phi_+(z)^{-1} = \sum_{j=0}^{\infty} z^j \gamma_j^+ \qquad (z \in \mathbb{T}).$$

Now, assume that the matrix symbol Φ is rational, i.e., its entries are quotients of scalar polynomials. Then one may use realization theorems from mathematical systems theory (see [K]) to show that Φ admits a representation of the form

$$\Phi(z) = I + C(zG - A)^{-1}B, \qquad z \in \mathbb{T}, \tag{0.3}$$

where A and G are square matrices of which the order n may be larger than the order m of Φ, the pencil $zG - A$ is regular on the unit circle $|z| = 1$, i.e., $\det(zG - A) \neq 0$ for $|z| = 1$, and the matrices B and C have sizes $n \times m$ and $m \times n$, respectively (see [GK], Theorem 3.1). The results about block Toeplitz operators summarized above can be reformulated in terms of the representation (0.3). In fact, the following theorems hold (see Sections 5, 6 and 9 in [GK]).

THEOREM 0.1. *Let T_Φ be the block Toeplitz operator on ℓ_m^2 with rational matrix symbol Φ given by* (0.3). *Put $A^\times = A - BC$. Then T_Φ is invertible if and only if the following two conditions hold*:

(α) $\det(zG - A^\times) \neq 0$ *for* $|z| = 1$,

(β) $\mathbb{C}^n = \operatorname{Im} P \oplus \operatorname{Ker} P^\times$,

where n is the order of the matrices G and A, and

$$P = \frac{1}{2\pi i} \int_{\mathbb{T}} G(\zeta G - A)^{-1} d\zeta, \qquad P^\times = \frac{1}{2\pi i} \int_{\mathbb{T}} G(\zeta G - A^\times)^{-1} d\zeta. \tag{0.4}$$

In this case the inverse of T_Φ is obtained in the following. Put

$$E^\times = \frac{1}{2\pi i} \int_{\mathbb{T}} (1 - \zeta^{-1})(\zeta G - A^\times)^{-1} d\zeta,$$

$$\Omega^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \zeta^{-1})G(\zeta G - A^\times)^{-1} d\zeta.$$

Then the entries of the inverse $T_\Phi^{-1} = [\Gamma_{ij}]_{i,j=0}^{\infty}$ *are given by*

$$\Gamma_{ij} = \Phi_{i-j}^\times + K_{ij}, \qquad i,j = 0,1,2,\ldots,$$

$$\Phi_k^\times = \begin{cases} CE^\times(\Omega^\times)^k(I - P^\times)B, & k = 1,2,\ldots, \\ I + CE^\times(I - P^\times)B, & k = 0, \\ -CE^\times(\Omega^\times)^{-k-1}P^\times B, & k = -1,-2,\ldots, \end{cases}$$

$$K_{ij} = CE^\times(\Omega^\times)^i(I - P^\times)\rho P^\times(\Omega^\times)^j B,$$

where ρ *is the projection of* $\mathbb{C}^n$ *along* $\operatorname{Im} P$ *onto* $\operatorname{Ker} P^\times$.

THEOREM 0.2. *Let* T_Φ *be the block Toeplitz operator on* ℓ_m^2 *with rational matrix symbol* Φ *given by* (0.3). *Put* $A^\times = A - BC$. *Then* T_Φ *is a Fredholm operator if and only if* $\det(zG - A^\times) \neq 0$ *for* $|z| = 1$. *In this case,*

$$\operatorname{index} T_\Phi = \operatorname{rank} P - \operatorname{rank} P^\times,$$

where P *and* $P^\times$ *are defined by* (0.4).

THEOREM 0.3. *Let* Φ *be a rational* $m \times m$ *matrix function given by* (0.3). *Put* $A^\times = A - BC$. *Then* Φ *admits a right canonical factorization relative to* $\mathbb{T}$ *if and only if the following two conditions hold*:

(i) $\det(zG - A^\times) \neq 0$ *for* $|z| = 1$,

(ii) $\mathbb{C}^n = \operatorname{Im} Q \oplus \operatorname{Ker} Q^\times$ *and* $\mathbb{C}^n = \operatorname{Im} P \oplus \operatorname{Ker} P^\times$.

Here n *is the order of the matrices* G *and* A, *and*

$$\begin{aligned} Q &= \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)^{-1} G d\zeta, \quad P = \frac{1}{2\pi i}\int_{\mathbb{T}} G(\zeta G - A)^{-1} d\zeta, \\ Q^\times &= \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A^\times)^{-1} G d\zeta, \quad P^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} G(\zeta G - A^\times)^{-1} d\zeta. \end{aligned} \tag{0.5}$$

In that case a right canonical factorization $\Phi(\zeta) = \Phi_-(\zeta)\Phi_+(\zeta)$ *of* Φ *relative to* $\mathbb{T}$ *is obtained by taking*

$$\Phi_-(\zeta) = I + C(\zeta G - A)^{-1}(I - \rho)B, \qquad \zeta \in \mathbb{T},$$

$$\Phi_+(\zeta) = I + C\tau(\zeta G - A)^{-1}B, \qquad \zeta \in \mathbb{T},$$

$$\Phi_-(\zeta)^{-1} = I - C(I-\tau)(\zeta G - A^\times)^{-1}B, \qquad \zeta \in \mathbb{T},$$

$$\Phi_+(\zeta)^{-1} = I - C(\zeta G - A^\times)^{-1}\rho B, \qquad \zeta \in \mathbb{T}.$$

Here τ is the projection of $\mathbb{C}^n$ along $\operatorname{Im} Q$ *onto* $\operatorname{Ker} Q^\times$ *and ρ is the projection along* $\operatorname{Im} P$ *onto* $\operatorname{Ker} P^\times$. *Furthermore, the two equalities in* (ii) *are not independent; in fact, the first equality in* (ii) *implies the second and conversely.*

The aim of the present paper is to generalize Theorems 0.1–0.3 to a natural class of non-Toeplitz operators. First, let us remark that the representation (0.3) allows us to view the corresponding block Toeplitz operator T_Φ as the input-output operator of the following discrete time system:

$$\text{(0.6)} \qquad \begin{cases} Ax_{k+1} = Gx_k + Bu_k & (k = 0, 1, \dots) \\ y_k = -Cx_{k+1} + u_k & (k = 0, 1, \dots) \\ x_0 \in \operatorname{Im} Q \end{cases}$$

where Q is the generalized Riesz projection appearing in (0.5). Such a representation appears in [GK]. The above fact gives a hint for the class of non-Toeplitz operators that will be considered. To be more specific, we shall deal here with non-Toeplitz operators $T = [T_{ij}]_{i,j=0}^\infty$ that appear as input-output operators of time-varying discrete time systems. The role of the projection Q in (0.6), and of the projections P, $P^\times$ and $Q^\times$ in Theorems 0.1–0.3 is taken over by dichotomies for certain difference equations.

This paper consists of five sections (not counting the present introduction). In the first section we recall the notion of a dichotomy and some of its properties. The second section gives an intrinsic characterization of the class of operators that we are dealing with. The time-varying analogues of Theorems 0.1–0.3 appear in Sections 3–5, respectively. The present paper has the character of a research announcement; full proofs will appear in [BGK2] and [BGK3].

1. PRELIMINARIES ABOUT DICHOTOMY

We begin by defining the notion of a dichotomy. Let a system

$$\text{(1.1)} \qquad A_{k+1}x_{k+1} = G_k x_k \qquad (k = 0, 1, \dots),$$

be given, where $(A_{k+1})_{k=0}^\infty$ and $(G_k)_{k=0}^\infty$ are bounded sequences of $n \times n$ matrices. We consider bounded sequences of projections $(I - Q_k)_{k=0}^\infty$ in $\mathbb{C}^n$ satisfying the following conditions:

$$\text{(1.2)} \qquad \operatorname{rank} Q_k \text{ is constant,}$$

$$\text{(1.3)} \qquad \operatorname{Im}(G_k|_{Ker\,Q_k}) \subset \operatorname{Im}(A_{k+1}|_{Ker\,Q_{k+1}}),$$

$$\text{(1.4)} \qquad \operatorname{Im}(A_{k+1}|_{\operatorname{Im}Q_{k+1}}) \subset \operatorname{Im}(G_k|_{\operatorname{Im}Q_k}),$$

for $k = 0, 1, \ldots$. Here I stands for the identity operator on $\mathbb{C}^n$. We a[illegible]nsider the abstract direct sum $\operatorname{Ker} Q_{k+1} \oplus \operatorname{Im} Q_k$, with $\left\|\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\|^2 = (\|u\|^2 + \|v\|^2)^{1/2}$, fo[illegible]Ker Q_{k+1}, $v \in \operatorname{Im} Q_k$, and assume that the mappings

$$[A_{k+1}|_{Ker\,Q_{k+1}}, G_k|_{\operatorname{Im}G_k}] \colon \operatorname{Ker} Q_{k+1} \oplus \operatorname{Im} Q_k \to \mathbb{C}^n \qquad (k = 0, 1, \ldots),$$

given by

$$[A_{k+1}|_{Ker\,Q_{k+1}}, G_k|_{\operatorname{Im}Q_k}] \begin{bmatrix} u \\ v \end{bmatrix} = A_{k+1}u + G_k v$$

for $u \in \operatorname{Ker} Q_{k+1}$, $v \in \operatorname{Im} Q_k$, are invertible with

$$\text{(1.5)} \qquad \sup_{k \geq 0} \|[A_{k+1}|_{Ker\,Q_{k+1}}, G_k|_{\operatorname{Im}Q_k}]^{-1}\| < \infty.$$

A bounded sequence of projections $(I - Q_k)_{k=0}^{\infty}$ with the properties (1.2)–(1.5) above is called a *dichotomy* for the system (1.1) if there exist positive constants a and M, with $a < 1$, such that

$$\text{(1.6)} \qquad \|(A_{k+j}|_{Ker\,Q_{k+j}})^{-1} G_{k+j-1} \cdots (A_{k+1}|_{Ker\,Q_{k+1}})^{-1} G_k (I - Q_k)\| \leq M a^j,$$

$$\text{(1.7)} \qquad \|(G_k|_{\operatorname{Im}Q_k})^{-1} A_{k+1} \cdots (G_{k+j-1}|_{\operatorname{Im}Q_{k+j-1}})^{-1} A_{k+j} Q_{k+j}\| \leq M a^j,$$

for $j, k = 0, 1, \ldots$. In this case the constant $\operatorname{rank}(I - Q_k)$ is called the *rank of the dichotomy*.

Note that by (1.2) the invertibility of $[A_{k+1}|_{Ker\,Q_{k+1}}, G_k|_{\operatorname{Im}Q_k}]$ for $k = 0, 1, \ldots$, is equivalent to the invertibility of the mappings

$$\text{(1.8)} \qquad G_k|_{\operatorname{Im}Q_k} \colon \operatorname{Im} Q_k \to \operatorname{Im}(G_k|_{\operatorname{Im}Q_k}), \qquad k = 0, 1, \ldots$$

$$(1.9) \qquad A_{k+1}|_{Ker\, Q_{k+1}}\colon \operatorname{Ker} Q_{k+1} \to \operatorname{Im}(A_{k+1}|_{Ker\, Q_{k+1}}), \qquad k = 0, 1, \ldots$$

and to the existence of a projection P_k in $\mathbb{C}^n$ satisfying

$$(1.10a) \qquad \operatorname{Im} P_k = \operatorname{Im}(G_k|_{\operatorname{Im} Q_k}), \qquad k = 0, 1, \ldots$$

$$(1.10b) \qquad \operatorname{Ker} P_k = \operatorname{Im}(A_{k+1}|_{Ker\, Q_{k+1}}), \qquad k = 0, 1, \ldots .$$

We call $(P_k)_{k=0}^{\infty}$ the *dual sequence* of projections of the dichotomy $(I - Q_k)_{k=0}^{\infty}$. In particular, since the mappings in (1.8)–(1.9) are invertible, it follows from inclusions (1.3)–(1.4) that the products in the inequalities (1.6)–(1.7) are well defined. The inequality (1.5) is equivalent to

$$(1.11) \qquad \sup_k \left(\|(A_{k+1}|_{Ker\, Q_{k+1}})^{-1}\|, \|(G_k|_{\operatorname{Im} Q_k})^{-1}\|, \|P_k\| \right) < \infty.$$

This definition of dichotomy of singular systems appears in [BG] and [BGK1], where it is called normal dichotomy.

Let us mention two special cases that are of particular interest. We say that the system (1.1) is *dichotomically regular* if

$$(1.12a) \qquad A_{k+1} = I, \quad Q_k = 0 \qquad (k = 0, 1, \ldots)$$

$$(1.12b) \qquad \lim_{j\to\infty} \sup \left(\sup_{k\geq 0} \|G_{k+j-1} \cdots G_k\|^{1/j} \right) < 1.$$

In this case, conditions (1.2)–(1.5) are fulfilled trivially, condition (1.7) is vacuous, and (1.6) is equivalent to (1.12b). Thus a dichotomically regular system has dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $Q_k = 0$ for each k.

We say that the system (1.1) is *dichotomically coregular* if

$$(1.13a) \qquad G_k = I, \quad Q_k = I \qquad (k = 0, 1, \ldots),$$

$$(1.13b) \qquad \lim_{j\to\infty} \sup \left(\sup_{k\geq 0} \|A_{k+1} \cdots A_{k+j}\|^{1/j} \right) < 1.$$

Also in this case, conditions (1.2)-(1.5) are fulfilled trivially. Now condition (1.6) is vacuous, and (1.7) is equivalent to (1.13b). In particular, a dichotomically coregular system is a dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $Q_k = I$.

A system may have different dichotomies. Theorems 1.1 and 1.2 below (which appear, respectively, as Corollary 6.5 in [BG] and as part of Theorem 1.2 in [BGK1]) describe the freedom one has in the choice of the dichotomies.

THEOREM 1.1. *If the system* (1.1) *admits a dichotomy* $(I - Q_k)_{k=0}^{\infty}$, *then for* $k = 0, 1, \dots$

$$\operatorname{Ker} Q_k = \{x_k \in \mathbb{C}^n : \exists x_{k+1}, x_{k+2}, \dots \text{ in } \mathbb{C}^n \text{ such that } A_{n+1}x_{n+1} = G_n x_n \ (n \geq k) \text{ and } \lim_{n\to\infty} x_n = 0\}.$$

In particular, $\operatorname{Ker} Q_k$ *and* $\operatorname{Ker} P_k = \operatorname{Im}(A_{k+1}|_{\operatorname{Ker} Q_{k+1}})$ *are uniquely defined, and all the dichotomies of* (1.1) *have the same rank.*

THEOREM 1.2. *If the system* (1.1) *admits a dichotomy* $(I - Q_k)_{k=0}^{\infty}$, *then for each subspace* L *of* $\mathbb{C}^n$ *with* $L \oplus \operatorname{Ker} Q_0 = \mathbb{C}^n$, *there exists a unique dichotomy* $(I - \overline{Q}_k)_{k=0}^{\infty}$ *of* (1.1) *with* $\operatorname{Im} \overline{Q}_0 = L$. *Furthermore, all the dichotomies of* (2.1) *are obtained in this way.*

It will be convenient to consider two types of operations on systems of the form (1.1). Consider a second system

$$\tilde{A}_{k+1}x_{k+1} = \tilde{G}_k x_k, \qquad k = 0, 1, \dots, \tag{1.14}$$

where $(\tilde{A}_{k+1})_{k=0}^{\infty}$ and $(\tilde{G}_k)_{k=0}^{\infty}$ are bounded sequences of $\tilde{n} \times \tilde{n}$ matrices. The systems (1.1) and (1.14) are said to be *equivalent* if $n = \tilde{n}$ and there exist invertible $n \times n$ matrices E_k and F_k, $k = 0, 1, \dots$ such that

$$\sup_{k\geq 0}\{\|E_k^{\pm 1}\|, \|F_k^{\pm 1}\|\} < \infty, \tag{1.15a}$$

$$\tilde{A}_{k+1} = F_k^{-1} A_{k+1} E_{k+1}, \quad \tilde{G}_k = F_k^{-1} G_k E_k \qquad (k = 0, 1, \dots). \tag{1.15b}$$

In this case a sequence of projections $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy of (1.1) if and only if $(I - E_k^{-1} Q_k E_k)_{k=0}^{\infty}$ is a dichotomy of (1.14).

The second operation is that of forming direct sums. By definition, the *direct sum* of the systems (1.1) and (1.14) is the system

$$\begin{bmatrix} A_{k+1} & 0 \\ 0 & \tilde{A}_{k+1} \end{bmatrix} x_{k+1} = \begin{bmatrix} G_k & 0 \\ 0 & \tilde{G}_k \end{bmatrix} x_k, \qquad k = 0, 1, \dots. \tag{1.16}$$

If $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy of (1.1) and $(I - \tilde{Q}_k)_{k=0}^{\infty}$ is a dichotomy of (1.14), then it is straightforward to check that the sequence of projections $(I - \Pi_k)_{k=0}^{\infty}$ where

$$\Pi_k = \begin{pmatrix} Q_k & 0 \\ 0 & \tilde{Q}_k \end{pmatrix} \qquad k = 0, 1, \ldots,$$

is a dichotomy of the direct sum (1.16).

THEOREM 1.3. *In order that the system* (1.1) *has a dichotomy it is necessary and sufficient that* (1.1) *is equivalent to a direct sum of a dichotomically regular and a dichotomically coregular system.*

We conclude this section with some relations with operator theory. Consider the system (1.1), and let L be a subspace of $\mathbb{C}^n$. By ℓ_n^2 we denote the Hilbert space of all norm square summable sequences with entries in $\mathbb{C}^n$, and

$$\ell_{n,L}^2 = \{(x_0, x_1, \ldots) \in \ell_n^2 \mid x_0 \in L\}. \tag{1.21}$$

We define two operators as follows:

$$G: \ell_{n,L}^2 \to \ell_n^2, \qquad G(x_0, x_1, \ldots) = (G_0x_0, G_1x_1, \ldots), \tag{1.22}$$

$$A: \ell_{n,L}^2 \to \ell_n^2, \qquad A(x_0, x_1, \ldots) = (A_1x_1, A_2x_2, \ldots). \tag{1.23}$$

The following result is contained in Theorem 1.1 and Proposition 2.3 of [BGK1].

THEOREM 1.4. *Let A and G be as* (1.22)–(1.23) *respectively. Then the operator $G - A$ is invertible if and only if the system* (1.1) *admits a unique dichotomy $(I - Q_k)_{k=0}^{\infty}$ with* $\operatorname{Im} Q_0 = L$. *Moreover, $\lambda G - A$ is invertible for each λ on the unit circle $\mathbb{T}$ if and only if it is invertible for one λ on the unit circle, and in this case*

$$\frac{1}{2\pi i}\int_{\mathbb{T}} (\lambda G - A)^{-1} G d\lambda = \operatorname{diag}(I|_L, Q_1, Q_2, \ldots),$$

and

$$\frac{1}{2\pi i}\int_{\mathbb{T}} G(\lambda G - A)^{-1} d\lambda = \operatorname{diag}(P_0, P_1, \ldots),$$

where $(I - Q_k)_{k=0}^{\infty}$ is the unique dichotomy of (1.1) *with* $\operatorname{Im} Q_0 = L$, *and $(P_k)_{k=0}^{\infty}$ is its dual sequence of projections.*

The next result gives an interpretation of the dichotomy in the time invariant case.

LEMMA 1.5. *Let A and G be $n \times n$ matrices. Then the system $Ax_{k+1} = Gx_k$ $(k = 0, 1, \ldots)$ admits a dichotomy if and only if $\lambda G - A$ is invertible for $|\lambda| = 1$. In this case, there exists a unique time invariant dichotomy $I - Q_k = I - Q$, where*

$$Q = \frac{1}{2\pi i} \int_{\mathbf{T}} (\lambda G - A)^{-1} G d\lambda,$$

and the dual sequence is given by $P_k = P$ $(k = 0, 1, \ldots)$, where

$$P = \frac{1}{2\pi i} \int_{\mathbf{T}} G(\lambda G - A)^{-1} d\lambda.$$

2. REALIZATION THEOREM

In this paper we are interested in operators that appear as input-output operators of an input-output system. The input-output systems that we have in mind are singular time-varying systems of the form

$$\Sigma \begin{cases} A_{k+1}x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L. \end{cases}$$

Here, $(G_k)_{k=0}^{\infty}$ and $(A_{k+1})_{k=0}^{\infty}$, $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ are bounded sequences of matrices of sizes $n \times n$, $n \times n$, $n \times m$, $m \times n$, respectively, and we assume that

$$(2.1) \qquad A_{k+1}x_{k+1} = G_k x_k \qquad (k = 0, 1, \ldots),$$

has a dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $\operatorname{Im} Q_0 = L$.

Choose an input sequence $(u_0, u_1, \ldots)$ in ℓ_m^2. Then, by Theorem 1.4, the first equation in Σ has a unique solution $(x_0, x_1, \ldots) \in \ell_{n,L}^2$. Inserting the latter sequence into the second equation in Σ yields an output sequence $(y_0, y_1, \ldots) \in \ell_m^2$, which is uniquely determined by $(u_0, u_1, \ldots)$. It follows that Σ has a well defined input-output operator, denoted by T_Σ, which acts as a bounded linear operator on ℓ_m^2. The latter statement also follows from Theorem 1.4 and from the fact that the sequences $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ are bounded.

As usual for operators on ℓ_m^2, we represent T_Σ by an infinite block matrix $T_\Sigma = [\tau_{ij}]_{ij=0}^{\infty}$, where each τ_{ij} is an $m \times m$ matrix. A straightforward application of Theorem 1.1 in [BGK1] shows that in this case

$$(2.2a) \qquad \begin{aligned} \tau_{ij} = & -C_{i+1}(A_{i+1}|_{Ker\, Q_{i+1}})^{-1} G_i \cdots (A_{j+2}|_{Ker\, Q_{j+2}})^{-1} G_{j+1} (A_{j+1}|_{Ker\, Q_{j+1}})^{-1} \\ & \cdot (I_n - P_j) B_j \qquad (i > j), \end{aligned}$$

$$(2.2b) \qquad \tau_{ij} = I_m - C_{i+1}(A_{i+1}|_{Ker\, Q_{i+1}})^{-1}(I_n - P_i)B_i \qquad (i = j),$$

$$(2.2c) \qquad \begin{aligned} \tau_{ij} = {} & C_{i+1}(G_{i+1}|_{\mathrm{Im}\, Q_{i+1}})^{-1} A_{i+2} \cdots (G_{j-1}|_{\mathrm{Im}\, Q_{j-1}})^{-1} \\ & \cdot A_j (G_j|_{\mathrm{Im} Q_j})^{-1} P_j B_j \qquad (i < j). \end{aligned}$$

Here $(P_k)_{k=0}^{\infty}$ is the dual sequence of projections of the dichotomy $(I - Q_k)_{k=0}^{\infty}$. Since the sequence $(I - Q_k)_{k=0}^{\infty}$ is a dichotomy, we can use the boundedness of the sequences $(B_k)_{k=0}^{\infty}$ and $(C_{k+1})_{k=0}^{\infty}$ and the estimates (1.6), (1.7) and (1.11) to show that there exist constants $M > 0$, $0 < a < 1$, such that

$$(2.3) \qquad \|\tau_{ij}\| \leq M a^{|i-j|} \qquad (i, j = 0, 1, \ldots).$$

In the next three sections, we study the invertibility of the operator T_Σ, its Fredholm properties and its UL-factorizations. In the present section we characterize the class of operators T on ℓ_m^2 that appear as input-output operators T_Σ of the type considered here.

Consider a bounded linear operator

$$(2.4) \qquad T = [t_{ij}]_{ij=0}^{\infty} : \ell_m^2 \to \ell_m^2.$$

We say that T admits a *realization* if $T = T_\Sigma$ for some input-output system Σ of the form described in the first paragraph of this section.

The following result holds.

THEOREM 2.1. *Let $T = [t_{ij}]_{ij=0}^{\infty}$ be a bounded linear operator in ℓ_m^2. Then T admits a realization if and only if*

$$(2.5) \qquad \|t_{ij}\| \leq M a^{|i-j|} \qquad (i, j = 0, 1, \ldots)$$

for some positive constants M, a with $a < 1$, and

$$(2.6) \qquad \sup_{\nu = 0, 1, \ldots} \{\operatorname{rank} H_\nu^-, \operatorname{rank} H_\nu^+\} < \infty,$$

where

$$H_\nu^- = \begin{bmatrix} t_{\nu 0} & t_{\nu 1} & \cdots & t_{\nu\nu} \\ t_{\nu+1,0} & t_{\nu+1,1} & \cdots & t_{\nu+1,\nu} \\ \vdots & \vdots & & \vdots \end{bmatrix}, \quad H_\nu^+ = \begin{bmatrix} t_{0,\nu} & t_{0,\nu+1} & \cdots \\ \vdots & \vdots & \\ t_{\nu,\nu} & t_{\nu,\nu+1} & \cdots \end{bmatrix} \qquad (\nu = 0, 1, \ldots).$$

If conditions (2.5) and (2.6) are fulfilled, then one may construct a realization of T explicitly in terms of the matrices H_ν^- and H_ν^+ by using restrictions of forward and backward shift operators acting on appropriate sequence spaces. Theorem 2.1 may be viewed as an operator version of the realization theorems in Section 5 of [GKLe], which are algebraic in nature and concern lower triangular block matrices which do not have to be related to bounded operators on an ℓ_2-space.

Finally, we make one remark about band operators. An operator $T = [t_{ij}]_{ij=0}^{\infty}$ in ℓ_m^2 is called a band operator if there exists a positive integer r such that $t_{ij} = 0$ whenever $|i-j| > r$. Theorem 2.1 shows that each band operator admits a realization. The proof of Theorem 2.1 will appear in [BGK3].

3. INVERTIBILITY

In this section we describe the invertibility properties of input-output operators in terms of dichotomies of systems.

We let T be the input-output operator of

$$\Sigma \begin{cases} A_{k+1}x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ \quad y_k = -C_{k+1}x_{k+1} + u_k & (k = 0, 1, \ldots), \\ \quad\quad x_0 \in L. \end{cases}$$

The description of T appears in the previous section, and Theorem 2.1 describes the class of operators T which are obtained in this way.

As in the previous section we assume that the system

$$A_{k+1}x_{k+1} = G_k x_k \qquad (k = 0, 1, \ldots), \tag{3.1}$$

admits a unique dichotomy $(I - Q_k)_{k=0}^{\infty}$ with

$$\operatorname{Im} Q_0 = L. \tag{3.2}$$

The next result shows that the invertibility properties of T are determined by a dichotomy property of the following *associated system*

$$A_{k+1}^{\times}x_{k+1} = G_k x_k \qquad (k = 0, 1, \ldots), \tag{3.3}$$

where

$$A_{k+1}^{\times} = A_{k+1} - B_k C_{k+1} \qquad (k = 0, 1, \ldots). \tag{3.4}$$

THEOREM 3.1. *Let T be the input-output operator of the system Σ, let $(I - Q_k)_{k=0}^{\infty}$ be the unique dichotomy of (3.1) satisfying (3.2), and denote by $(P_k)_{k=0}^{\infty}$ the corresponding dual sequence of projections. Then the following conditions are equivalent:*

(I) *The operator T is invertible in ℓ_m^2.*

(II) *The associated system (3.3) admits a dichotomy $(I - Q_k^{\times})_{k=0}^{\infty}$ such that $\operatorname{Im} Q_0 \oplus \operatorname{Ker} Q_0^{\times} = \mathbb{C}^n$.*

(III) *The associated system (3.4) admits a dichotomy whose dual sequence $(P_k^{\times})_{k=0}^{\infty}$ satisfies $\operatorname{Im} P_0 \oplus \operatorname{Ker} P_0^{\times} = \mathbb{C}^n$.*

Moreover, if T is invertible, then T^{-1} is the input-output operator of the system

$$\Sigma^{\times} \begin{cases} A_{k+1}^{\times} x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots), \\ y_k = C_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L. \end{cases}$$

For a smaller class of operators involving regular input-output sytems and a stronger notion of dichotomy Theorem 3.1 appears in [GKvS]; the proof of Theorem 3.1 will appear in [BGK3].

Let us also remark that using the explicit description of an input-output operator, one can give formulas for the entries of T^{-1} in the following way. Let $(I - Q_k^{\times})_{k=0}^{\infty}$ be an arbitrary dichotomy of (3.3) with dual sequence of projections $(P_k^{\times})_{k=0}^{\infty}$. Then the entries Γ_{ij} of $T^{-1} = (\Gamma_{ij})_{ij=0}^{\infty}$, where T is the input-output operator of Σ, are given by $\Gamma_{ij} = \Phi_{ij}^{\times} + K_{ij}$ where

$$\begin{aligned} \Phi_{ij}^{\times} = {} & C_{i+1}(A_{i+1}^{\times}|_{Ker\,Q_{i+1}^{\times}})^{-1} G_i \cdots (A_{j+2}^{\times}|_{Ker\,Q_{j+2}^{\times}})^{-1} G_{j+1} (A_{j+1}^{\times}|_{Ker\,Q_{j+1}^{\times}})^{-1} \\ & \cdot (I - P_j^{\times}) B_j \qquad (i > j), \end{aligned} \tag{3.5a}$$

$$\Phi_{ij}^{\times} = I_m + C_{ij}(A_{i+1}^{\times}|_{Ker\,Q_{i+1}^{\times}})^{-1}(I - P_i^{\times}) B_i \qquad (i = j), \tag{3.5b}$$

$$\begin{aligned} \Phi_{ij} = {} & -C_{i+1}(G_{i+1}|_{\operatorname{Im} Q_{i+1}^{\times}})^{-1} A_{i+1}^{\times} \cdots (G_{j-1}|_{\operatorname{Im} Q_{j-1}^{\times}})^{-1} \\ & \cdot A_j^{\times}(G_j|_{\operatorname{Im} Q_j^{\times}})^{-1} P_j^{\times} B_j \qquad (i < j), \end{aligned} \tag{3.5c}$$

and

$$(3.6)\qquad \begin{aligned} K_{ij} = {} & C_{i+1}(A_{i+1}^{\times}|_{Ker\,Q_{i+1}^{\times}})^{-1}G_i\cdots(A_2^{\times}|_{Ker\,Q_2^{\times}})^{-1}G_1(A_1^{\times}|_{Ker\,Q_1^{\times}})^{-1}\rho \\ & \cdot A_1^{\times}(G_1|_{\mathrm{Im}\,Q_1^{\times}})^{-1}\cdots A_j^{\times}(G_j|_{\mathrm{Im}Q_j^{\times}})^{-1}P_j^{\times}B_j \qquad (i,j=0,1,\ldots), \end{aligned}$$

where ρ is the projection of $\mathbb{C}^n$ along $\operatorname{Im} P_0$ onto $\operatorname{Ker} P_0^{\times}$.

In the case when $T = T_{\Phi}$ is a Toeplitz operator with rational matrix symbol $\Phi(z) = I + C(zG - A)^{-1}B$ ($z \in \mathbb{T}$), it is easy to see that Theorem 0.1 of the introduction follows from Theorem 3.1 and equalities (3.5)-(3.6) above, and Lemma 1.5.

4. FREDHOLM PROPERTIES

The Fredholm properties of input-output operators are described using the notion of an asymptotic dichotomy. We say that the system $A_{k+1}x_{k+1} = G_kx_k$ ($k = 0,1,\ldots$) admits an *asymptotic dichotomy* if there exists a nonnegative integer N such that the system $A_{k+N+1}x_{k+1} = G_{k+N}x_k$ ($k = 0,1,\ldots$) admits a dichotomy. It follows from Theorem 1.1 that the rank of a dichotomy of the latter system is independent of N. We call this common rank the *rank of the asymptotic dichotomy*.

As in the previous section, we consider the input-output operator T of a system

$$\Sigma \begin{cases} A_{k+1}x_{k+1} = G_kx_k + B_ku_k & (k = 0,1,\ldots), \\ y_k = -C_{k+1}x_{k+1} + u_k & (k = 0,1,\ldots), \\ x_0 \in L, \end{cases}$$

and we assume that the system $A_{k+1}x_{k+1} = G_kx_k$ ($k = 0,1,\ldots$) admits a dichotomy $(I - Q_k)_{k=0}^{\infty}$ with $\operatorname{Im} Q_0 = L$.

THEOREM 4.1. *Let T be the input-output operator of the system Σ above. Put $A_{k+1}^{\times} = A_{k+1} - B_kC_{k+1}$ ($k = 0,1,\ldots$). Then T is Fredholm if and only if the system*

$$(4.1)\qquad A_{k+1}^{\times}x_{k+1} = G_kx_k \qquad (k = 0,1,\ldots)$$

admits an asymptotic dichotomy, and in this case

$$(4.2)\qquad \operatorname{index} T = p + \dim L - n,$$

where p denotes the rank of an asymptotic dichotomy of (4.1).

It is easy to obtain Theorem 0.2 of the introduction from this result and Lemma 1.5. For a smaller class of operators involving regular input-output sytems and a stronger notion of dichotomy Theorem 4.1 appears in [GKvS]; the proof of Theorem 4.1 will appear in [BGK3].

5. CANONICAL FACTORIZATION

Let $T = [t_{ij}]_{ij=0}^{\infty}$ be a bounded operator in ℓ_m^2 in its standard matrix representation where t_{ij} are $m \times m$ matrices. The operator T is upper triangular (respectively lower triangular) if $t_{ij} = 0$ whenever $i > j$ (respectively $i < j$). We say that T is diagonal if $t_{ij} = 0$ whenever $i \neq j$. Let us remark that if T is upper (respectively lower) triangular and invertible, then T^{-1} is also upper (respectively lower) triangular. We say that T admits a *canonical upper lower factorization* if there exist an invertible upper triangular operator T_- and an invertible lower triangular operator T_+ such that $T = T_-T_+$. We refer to [GF] for canonical factorizations of operators and functions.

A necessary condition for T to admit canonical factorization is that T is invertible. This condition is not sufficient in general.

It is easily seen that if T admits two canonical upper lower factorizations $T = T_-T_+ = T'_-T'_+$, then there exist an invertible diagonal operator D such that $T'_- = T_-D$ and $T'_+ = D^{-1}T_+$. Conversely, if $T = T_-T_+$ is a canonical upper lower factorization for T, then defining T'_- and T'_+ by the above formulas we obtain another canonical factorization.

Assume that as in the introduction, $T = T_\Phi = [\Phi_{i-j}]_{ij=0}^{\infty}$ is a Toeplitz operator where Φ_k are $m \times m$ matrices with $\sum_{k=-\infty}^{\infty} \|\Phi_k\| < \infty$. Let $\Phi(z) = \sum_{k=-\infty}^{\infty} \Phi_k z^k$ $(z \in \mathbb{T})$ be the symbol of T. Assume that Φ admits a right canonical factorization $\Phi(z) = \Phi_-(z)\Phi_+(z)$, where Φ_+ and $\widetilde{\Phi}_-$, $\widetilde{\Phi}_-(z) = \Phi_-(z^{-1})$, are $m \times m$ matrix functions which are analytic on the open unit disc $\mathbb{D}$, continuous on $\mathbb{D} \cup \mathbb{T}$ and their determinants do not vanish on $\mathbb{D} \cup \mathbb{T}$. Put $\Phi_+(z) = \sum_{k=0}^{\infty} \gamma_k^+ z^k$, $\Phi_-(z) = \sum_{k=-\infty}^{0} \gamma_k^- z^k$, and set $\gamma_{-1}^+ = \gamma_{-2}^+ = \cdots = 0$ and $\gamma_1^- = \gamma_2^- = \cdots = 0$. Then the operators $T_{\Phi_+} = [\gamma_{i-j}^+]_{ij=0}^{\infty}$ and $T_{\Phi_-} = [\gamma_{i-j}^-]_{ij=0}^{\infty}$ are lower and upper triangular invertible operators, and the following canonical upper lower factorization holds

$$T_\Phi = T_{\Phi_-} T_{\Phi_+}. \tag{5.1}$$

Conversely if $T_\Phi = T_-T_+$ is a canonical upper lower factorization, then in particular,

T_Φ is invertible. Whence, Φ admits a right canonical factorization $\Phi = \Phi_- \Phi_+$. This factorization also induces the canonical upper lower factorization (5.1). By the uniqueness of the canonical upper lower factorization, it follows that there exists a diagonal operator D such that $T_- = T_{\Phi_-} D$ and $T_+ = D^{-1} T_{\Phi_+}$.

The above remark shows the equivalence between canonical upper lower factorization of Toeplitz operators, and right canonical factorization of matrix valued functions in the Wiener class. By this equivalence, Theorem 0.3 is equivalent to a statement about canonical upper lower factorization of Toeplitz operators with rational symbol.

We now present our extension of this result for input-output operators of time varying systems.

Let T be the input-output operator of the system

$$\Sigma \begin{cases} A_{k+1} x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ y_k = -C_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

where the system

$$A_{k+1} x_{k+1} = G_k x_k \qquad (k = 0, 1, \ldots) \tag{5.2}$$

admits a unique dichotomy $(I - Q_k)_{k=0}^\infty$ with

$$\operatorname{Im} Q_0 = L. \tag{5.3}$$

The next result gives necessary and sufficient conditions for the existence of a canonical upper lower factorization for T. The conditions are in terms of the associated system

$$A_{k+1}^\times x_{k+1} = G_k x_k \qquad (k = 0, 1, \ldots), \tag{5.4}$$

where $A_{k+1}^\times = A_{k+1} - B_k C_{k+1}$ $(k = 0, 1, \ldots)$.

In the statement below we use the following terminology about direct sums. Let be given a sequence of direct sum decompositions

$$V_k \oplus W_k = \mathbb{C}^n \qquad (k = 0, 1, \ldots). \tag{5.5}$$

We say that the direct sum decomposition (5.5) *holds uniformly* if the sequence of projections $(\Pi_k)_{k=0}^\infty$ defined via $\operatorname{Ker} \Pi_k = V_k$, $\operatorname{Im} \Pi_k = W_k$ $(k = 0, 1, \ldots)$, satisfies $\sup_{k \geq 0} \|\Pi_k\| < \infty$.

THEOREM 5.1. *Let T be the input-output operator of the system Σ, where the system* (5.2) *admits the dichotomy* $(I - Q_k)_{k=0}^{\infty}$ *with* (5.3), *and let* $(P_k)_{k=0}^{\infty}$ *be the corresponding dual sequence of projections. Then the following conditions are equivalent*:

(I) *The operator T admits a canonical upper lower factorization.*

(II) *The associated system* (5.4) *admits a dichotomy* $(I - Q_k^{\times})_{k=0}^{\infty}$ *such that the following direct sum holds uniformly*

$$\operatorname{Im} Q_k \oplus \operatorname{Ker} Q_k^{\times} = \mathbb{C}^n \qquad (k = 0, 1, \ldots). \tag{5.6}$$

(III) *The associated system* (5.4) *admits a dichotomy whose dual sequence of projections* $(P_k^{\times})_{k=0}^{\infty}$ *satisfies the following direct sum condition uniformly*

$$\operatorname{Im} P_k \oplus \operatorname{Ker} P_k^{\times} = \mathbb{C}^n \qquad (k = 0, 1, \ldots). \tag{5.7}$$

Moreover, assume that (5.6) *or* (5.7) *hold for one dichotomy of* (5.4). *Then* (5.6) *and* (5.7) *hold for every dichotomy* $(I - Q_k^{\times})_{k=0}^{\infty}$ *of* (5.4), *and a canonical upper lower factorization of T may be obtained in the following way. Let ρ_k and τ_k be the projections in $\mathbb{C}^n$ defined by*

$$\begin{aligned} \operatorname{Ker} \rho_k &= \operatorname{Im} P_k, \quad \operatorname{Im} \rho_k = \operatorname{Ker} P_k^{\times} \qquad (k = 0, 1, \ldots) \\ \operatorname{Ker} \tau_k &= \operatorname{Im} Q_k, \quad \operatorname{Im} \tau_k = \operatorname{Ker} Q_k^{\times} \qquad (k = 0, 1, \ldots). \end{aligned}$$

Then the canonical upper lower factorization $T = T_- T_+$ holds, where T_-, T_+, T_-^{-1}, T_+^{-1} are, respectively, the input-output operators of the following systems

$$\Sigma_- \begin{cases} A_{k+1} x_{k+1} = G_k x_k + (I - \rho_k) B_k u_k & (k = 0, 1, \ldots) \\ y_k = -C_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

$$\Sigma_+ \begin{cases} A_{k+1} x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ y_k = -C_{k+1} \tau_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

$$\Sigma_-^{\times} \begin{cases} A_{k+1}^{\times} x_{k+1} = G_k x_k + B_k u_k & (k = 0, 1, \ldots) \\ y_k = C_{k+1} (I - \tau_{k+1}) x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

and

$$\Sigma_+^\times \begin{cases} A_{k+1}^\times x_{k+1} = G_k x_k + \rho_k B_k u_k & (k = 0, 1, \ldots) \\ y_k = C_{k+1} x_{k+1} + u_k & (k = 0, 1, \ldots), \\ x_0 \in L, \end{cases}$$

where $A_{k+1}^\times = A_{k+1} - B_k C_{k+1}$ $(k = 0, 1, \ldots)$.

The proof of Theorem 5.1 will be given in [BGK2]. In contrast with the Toeplitz case, in the time-varying case the invertibility of T is not equivalent to the existence of a canonical upper lower factorization.

If $T = T_\Phi$ is a Toeplitz operator in ℓ_m^2 with a rational matrix valued symbol, then as in the preceding sections, one may represent Φ as in (0.3) and view T_Φ as the input-output operator of the system (0.6). Theorem 5.1 applies to this representation of T_Φ as an input-output operator. Using the description given in Lemma 1.5 of dichotomies of time invariant systems, it follows that condition (i) and either direct sum condition in (ii) of Theorem 0.3 are equivalent to condition (II) or condition (III) above. This proves the first part of Theorem 0.3. The explicit description of the right canonical factorization of Φ given in the second part of Theorem 0.3 follows from: (a) the connections between the right canonical factorization of Φ and the canonical upper lower factorization of the operator T_Φ, given in the beginning of this section, (b) the explicit formulas of the systems $\Sigma_\pm$ and $\Sigma_\pm^\times$ given in Theorem 5.1, and (c) the representation of Toeplitz operators as input-output systems, given by (0.6).

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Estimates for the $\bar{\partial}$-equation in the one-dimensional case

Bo Berndtsson

This talk is concerned with the equation

$$\frac{\partial u}{\partial \bar{z}} = f \tag{1}$$

in a domain Ω in $\mathbb{C}$. If φ is a subharmonic function in Ω, we are interested in estimates for (1) with the weight factor $e^{-\varphi}$, and in particular, we are interested in the canonical solution to (1) in $L^2(e^{-\varphi})$, meaning the solution of minimal norm in this space. These problems have previously been studied in a number of papers of which we refer to [F-Si], [Chr], [Be]. I will not make a systematic review of previous results, but rather make some additional remarks and formulate some open problems.

First we note that an estimate of u in terms of f in $L^2(e^{-\varphi})$ is equivalent to an estimate of $v = ue^{-\varphi/2}$ in terms of $g = fe^{-\varphi/2}$ in L^2. Making these substitutions, we can rewrite (1) as

$$\bar{D}_\varphi v = g \tag{2}$$

where $\bar{D}_\varphi = e^{-\varphi/2}\frac{\partial}{\partial \bar{z}}e^{\varphi/2}$. This substitution was used in [Chr] and it is quite helpful.

Next, we note that the formal adjoint of $\bar{D}_\varphi$ is $-D_\varphi$, where

$$D_\varphi = e^{\varphi/2}\frac{\partial}{\partial z}e^{-\varphi/2}.$$

The domain of the adjoint consists of functions that vanish on the boundary of Ω. Since the L^2-minimal solution must lie in the range of the adjoint, we are led to study the Dirichlet problem

$$\begin{aligned} \bar{D}_\varphi D_\varphi \alpha &= g \\ \alpha &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{3}$$

The canonical solution to (2) will then be given by

$$v = D_\varphi \alpha.$$

Now let us consider a number of kernels associated with the previous problems.

1. The Green's function $N_\varphi(z, \zeta)$ satisfies that

$$\alpha(z) = \int_\Omega N_\varphi(z, \zeta) g(\zeta) d\lambda(\zeta)$$

is the unique solution to (3).

2. The Poisson kernel $P_\varphi(\zeta, z)$ satisfies that

$$u(\zeta) = \int_{\partial\Omega} P_\varphi(\zeta, z) h(z)$$

is the unique solution to $\bar{D}_\varphi D_\varphi u = 0, u = h$ on $\partial\Omega$.

3. The kernel $K_\varphi(z, \zeta)$ satisfies that

$$v(z) = \int_\Omega K_\varphi(z, \zeta) g(\zeta) d\lambda(\zeta)$$

is the L^2-minimal solution to (2).

Using Green's formula, it is easy to verify that

$$P_\varphi(\zeta, z) = -i D_\varphi N_\varphi(z, \zeta) dz$$

when $z \in \partial\Omega$, just like in the case $\varphi = 0$. On the other hand, we also have

$$K_\varphi(z, \zeta) = D_\varphi N_\varphi(z, \zeta)$$

from the previous discussion, so we see that the behaviour of L^2-minimal solutions on the boundary is governed by the Poisson kernel, i.e., by the behaviour of functions that are "harmonic" with respect to the "Laplacian" $\bar{D}_\varphi D_\varphi$.

Lemma 1 *Let $\alpha \in \mathbb{C}^2$. Then*

$$\Delta|\alpha|^2 = 2Re(\bar{D}_\varphi D_\varphi \alpha)\bar{\alpha} + |D_\varphi \alpha|^2 + |\bar{D}_\varphi \alpha|^2 + \Delta\varphi|\alpha|^2$$

and

$$\Delta|\alpha| \geq -|\bar{D}_\varphi D_\varphi \alpha| + \frac{1}{2}\Delta\varphi|\alpha|$$

(where $\alpha \neq 0$).

In particular, we see that $|\alpha|$ is subharmonic if $\bar{D}_\varphi D_\varphi \alpha = 0$.

¿From this it follows that P_φ is dominated by the standard Poisson kernel, so we get:

Proposition 2 *Let φ be subharmonic. Then*

$$K_\varphi(z,\zeta)| \leq |K_0(z,\zeta)|$$

for $z \in \partial\Omega$.

This estimate is certainly false for $z \in \Omega$, but at least in the case when Ω is a disc, it holds in a weaker form, [Be].

We shall now see a case when the presence of a weight makes the estimates strictly better than in the unweighted case.

Proposition 3 *Let $\Omega = \{\rho < 0\}$ where ρ is subharmonic and $\rho \in C^1(\bar{\Omega})$. Let φ be subharmonic, and let v be the L^2-minimal solution to (2).*

Then it holds

$$\sup_{\partial\Omega} \frac{|v|}{|\partial\rho|} \leq \sup_{\Omega} \frac{|g|}{(-\rho)\Delta\varphi}.$$

When φ is bounded, Ω is the disc and $\rho = |z|^2 - 1$, this proposition implies the theorem that we get solutions to (1) that are bounded on the boundary if f is a Carleson measure. The proof is a consequence of the fact that if $\bar{D}_\varphi D_\varphi \alpha = 0$, then $|\alpha|$ is a subsolution to the Schrödinger operator $\Delta - \frac{1}{2}\Delta\varphi$. An interesting open problem is whether the conclusion of the analogous theorem of T. Wolff is also satisfied by a canonical solution in the same way.

Finally, we consider a related problem which probably requires a more detailed analysis of our operator $\bar{D}_\varphi D_\varphi$.

Conjecture *Let h be a bounded function on $\partial\Omega$ and let α_λ solve*

$$\begin{aligned} \bar{D}_\lambda \varphi D_{\lambda\varphi} \alpha_\lambda &= 0 \qquad in \quad \Omega \\ \alpha_\lambda &= h \qquad on \quad \partial\Omega. \end{aligned}$$

Assume that $\varphi \in C^\infty$, is subharmonic and satisfies

$$\{z \in \Omega, \Delta\varphi = 0\} \subset\subset \Omega.$$

Then

$$\overline{\lim}_{\lambda\to\infty} |\alpha_\lambda|^{1/\lambda} < 1$$

in Ω.

It is easy to see that

$$\overline{\lim}_{\lambda\to\infty} |\alpha_\lambda|^{1/\sqrt{\lambda}} < 1,$$

by using again that $|\alpha_\lambda|$ is a subsolution to the Schrödinger operator $\Delta - \frac{1}{2}\Delta\varphi$. The conjecture will follow if one can prove it when $\Delta\varphi > 0$ everywhere, and one may even assume that $\Delta\varphi \approx 1$. If, on the other hand, $\Delta\varphi = 1$, the conjecture holds. The reason for this is once again related to Schrödinger operators. The operator $\bar{D}_\varphi D_\varphi$ is *itself* a Schrödinger operator, but of a more general form than we have used so far – it includes a magnetic field $= \Delta\varphi$.

If this field is constant, one can compute solutions explicitly and verify the conjecture in that case.

The conjecture is related to the analysis of $\bar{\partial}_b$ on weakly pseudoconvex domains in $\mathbb{C}^2$, in particular to the problem of analytic hypoellipticity.

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NORMAL FORMS OF REAL SYMMETRIC SYSTEMS WITH MULTIPLICITY

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Dedicated to Jaap Korevaar on the occasion of his 70th birthday.

ABSTRACT. A normal form is given for real symmetric systems of linear partial differential equations, at points where the principal symbol has a two-dimensional kernel under assumptions which apply to the generic case.

1. Introduction

This paper takes a step towards understanding the propagation of polarization of solutions of real, symmetric linear systems, $\mathbf{Q}u = 0$, of partial differential equations. We will show that the operator $\mathbf{Q}$ can be brought into the very simple standard form

$$\begin{pmatrix} D_1 + D_2 & x_2 D_3 \\ x_2 D_3 & \pm(D_1 - D_2) \end{pmatrix}, \quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}. \tag{1.1}$$

This normal form can be achieved by splitting off elliptic summands, multiplying by invertible pseudodifferential operators and conjugating with invertible Fourier integral operators. The normal form is obtained modulo terms for which the full Taylor expansion of the principal symbol vanishes at every point of Σ. Here Σ is the subset of the cotangent bundle, where the principal symbol has a zero eigenvalue of multiplicity higher than one. The construction is microlocal, that is, in some conic neighborhood of a given point in the cotangent bundle. Of course, one would hope that the standard system is easier to investigate than the system in its original form.

In order to explain the assumptions under which the results can be proved, we recall that the propagation of polarization can be paraphrased mathematically as the behaviour of asymptotic high-frequency solutions of the m-th order system $\mathbf{Q}u = 0$. If

$$u(x) = e^{i\tau\langle x,\xi\rangle} a(x)$$

is a high frequency wave, with frequency τ, phase covector ξ and amplitude vector $a(x)$, then

$$(\mathbf{Q}u)(x) \sim \tau^m \, e^{i\tau\langle x, \xi\rangle} \, Q(x, \xi) \, a(x),$$

asymptotically as $\tau \to \infty$. Here $Q(x, \xi)$ is a matrix, which is called the *principal symbol* of the operator $\mathbf{Q}$, which is instrinsically defined on the *cotangent bundle* T^*M of M, the phase space of classical mechanics, on which (x, ξ) are canonical coordinates. One says that $u(x)$ is an *asymptotic solution* if $\mathbf{Q}(u)$ is of order τ^l with $l < m$.

The operator $\mathbf{Q}$ is called *elliptic at* (x, ξ) if $Q(x, \xi)$ is invertible. Clearly, $u(x)$ can only be an asymptotic solution if the amplitude vector $a(x)$ belongs to $\ker Q(x, \xi)$, the *polarization space* of $\mathbf{Q}$ at (x, ξ). Therefore, high-frequency solutions with nonzero amplitude vector can only occur if (x, ξ) lies in

$$N = \{(x, \xi) \in T^*M; \det Q(x, \xi) = 0\}$$

the *characteristic set* of $\mathbf{Q}$.

At the points of N where $\det Q$ has simple zeros, the polarization space is one-dimensional and one can reduce the study of the operator to that of the scalar case, cf. Dencker [8]. In turn the scalar case with simple zeroes can be reduced to the study of the operator $\frac{\partial}{\partial t}$, using multiplication by elliptic operators and conjugation by invertible Fourier integral operators, see Duistermaat and Hörmander [13, Sec. 6]. A generic scalar operator will only have these, so called, simple characteristics, see Nuij [22].

For effects which are truly specific for systems, we therefore must turn to the subset Σ of the points $(x, \xi) \in T^*M$, at which $\det Q$ has zeros with multiplicity more than one. This location is called the optical or acoustical axis by physicists when they consider the Maxwell equations or the equations for waves in elastic media. It has been known for some time that these multiplicities sometimes occur for topological reasons and are present generically, see Lax [19], John [14] and Hörmander [18].

Rather than investigating the situation for generic systems we shall assume that for each $(x, \xi) \in T^*M$, the principal symbol $Q(x, \xi)$ is a real and symmetric matrix. This is the case for many systems in mathematical physics, in particular for all systems arising from variational problems, even when arbitrary lower order terms are added as perturbations. Under explicit and generic nondegeneracy conditions, stated in section 3, we obtain the normal form in a conic neighborhood of $(x, \xi) \in \Sigma$.

The two sign choices in (1.1) lead to drastically different behaviour of the solutions. For the plus sign, the operator is hyperbolic with respect to the variable x_1. Close to Σ, the bicharacteristic curves in the regular part of the characteristic set N form helices, narrowly winding around smooth curves in Σ. Along with it, the polarization space rotates rapidly. For the minus sign, the operator is hyperbolic with respect to x_2. The bicharacteristic curves in N approach Σ and bounce away like a hyperbola approaching the intersection of its asymptotes. During the change of direction, the polarization space makes a quarter turn.

When thinking of systems with multiplicity, one first thinks of the phenomenon of conical refraction of light, in which a thin lightbeam changes into a cone of light upon reaching an bi-axial crystal. This was predicted by Hamilton and experimentally verified by Lloyd in 1837, see Hamilton [9] and Born and Wolf [5, p. xxii]. For studies with modern analytical tools see for instance Melrose and Uhlmann [21] and Dencker [7]. The normal forms for 2 by 2 systems with conical refraction are

$$\begin{pmatrix} D_1 + D_2 & D_3 \\ D_3 & D_1 - D_2 \end{pmatrix}.$$

The non-generic aspect of this system lies in the fact the it is independent of the coordinates in the base making the singular part of the characteristic variety involutive.

Our paper is organised as follows. In section 2 we discuss how splitting off elliptic factors leads to 2×2 systems. In section 3 we study some basic symplectic geometry of the symbol in order to formulate the assumptions under which the normal form can be achieved. In section 4 we state the result and begin the proof which we finish in section 5. We finish, in section 6, by checking that Maxwell's equations and the equations for elastodynamics satisfy the genericity assumptions and realise both signs.

This work was started subsequent to listening to a lecture delivered by V.I. Arnol'd in Utrecht in the spring of 1990. We thank him for his inspiration.

2. Splitting off elliptic summands

Let $E \to M$ be an k-dimensional smooth real vector bundle over an n-dimensional paracompact smooth manifold M. We will study a linear pseudodifferential operator $\mathbf{Q}$ of order m, acting on the space $\Gamma(M, E)$ of smooth sections over M. Since our constructions are (micro-)local, we use a local trivialization of E, in which $\mathbf{Q}$ can be identified with a $k \times k$-matrix of pseudodifferential operators. The principal symbol $Q(x, \xi)$ of $\mathbf{Q}$ is a $k \times k$-matrix, depending smoothly on $(x, \xi) \in T^*M \setminus 0$. It is homogeneous of degree m in the sense that

$$Q(x, \tau\xi) = \tau^m Q(x, \xi), \qquad \tau > 0.$$

The half-line $\{(x, \tau\xi) \mid \tau > 0\}$ is called the *cone axis* through (x, ξ), and subsets of T^*M are called conic when they are the union of cone axes. For example, the characteristic set N is a closed conic subset of $T^*M \setminus 0$, because $\det Q$ is homogeneous of degree km.

Near elliptic points, $\mathbf{Q}$ has a pseudodifferential parametrix $\mathbf{R} = \mathbf{Q}^{-1}$ of order $-m$, in the sense that $\mathbf{Q}\,\mathbf{R} \sim \mathbf{R}\,\mathbf{Q} \sim \mathbf{I}$ near (x, ξ). Here the expression "$\mathbf{A} \sim \mathbf{B}$ near (x, ξ)", for pseudodifferential operators $\mathbf{A}$, $\mathbf{B}$, means that $\mathbf{A} - \mathbf{B}$ is smoothing (has order $-\infty$) in a conic neighborhood of (x, ξ). We start with a basic, simple observation (compare Dencker [7, proof of Prop. 2.5] and Hörmander [18, Lemma 2.1]).

Lemma 1. *Suppose that we have the block decomposition*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{ll} & \mathbf{Q}_{lr} \\ \mathbf{Q}_{rl} & \mathbf{Q}_{rr} \end{pmatrix}. \tag{2.1}$$

in which $\mathbf{Q}_{ij}$ has i rows and j columns, $r = k - l$, and $\mathbf{Q}_{rr}$ is elliptic in a conic open subset U of $T^*M \setminus 0$. Then there exist elliptic pseudodifferential operators $\mathbf{A}$, $\mathbf{B}$ of order 0 such that in U

$$\mathbf{AQB} \sim \begin{pmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q}_{rr} \end{pmatrix}.$$

Here $\mathbf{P}^* - \mathbf{P}$ is of order $\leq \mu$ if $\mathbf{Q}^* - \mathbf{Q}$ is of order $\leq \mu$, the principal symbol $P(x, \xi)$ of $\mathbf{P}$ is real if the principal symbol $Q(x, \xi)$ of $\mathbf{Q}$ is real, and finally $P(x, \xi) = 0$ if $\dim \ker Q(x, \xi) = l$.

Proof. Take

$$\mathbf{A} \sim \begin{pmatrix} \mathbf{I} & -\mathbf{Q}_{lr}\mathbf{Q}_{rr}^{-1} \\ 0 & \mathbf{I} \end{pmatrix}, \quad \mathbf{B} \sim \begin{pmatrix} \mathbf{I} & 0 \\ -\mathbf{Q}_{rr}^{-1}\mathbf{Q}_{rl} & \mathbf{I} \end{pmatrix}.$$

Then we get the desired form for $\mathbf{AQB}$, with

$$\mathbf{P} \sim \mathbf{Q}_{ll} - \mathbf{Q}_{lr}\mathbf{Q}_{rr}^{-1}\mathbf{Q}_{rl}.$$

□

In the situation of Lemma 1 we will say that the $l \times l$ operator $\mathbf{P}$ is *split off* from the $k \times k$ operator $\mathbf{Q}$, and refer to the process as splitting off an elliptic summand after multiplication with elliptic factors. This process is compatible with the investigation of propagation of singularities. To see this, note that

$$u := \mathbf{B}\begin{pmatrix} v \\ w \end{pmatrix}$$

is smooth if and only if v and w are. Then $\mathbf{Q}u$ is smooth if and only if $\mathbf{P}v$ and w are smooth. In the context of asymptotic high-frequency solutions, "smooth" has to be replaced by "asymptotically small". Note also that the principal symbol $B(x, \xi)$ of $\mathbf{B}$ is an isomorphism from $\ker P(x, \xi) \oplus 0$ to the polarization space $\ker Q(x, \xi)$ of $\mathbf{Q}$. In particular these spaces have the same dimension.

We have

$$\det Q(x, \xi) = \det P(x, \xi)\, \det Q_{rr}(x, \xi),$$

in which $\det Q_{rr}(x, \xi)$ is pointwise non-zero. This implies that $\det Q$ and $\det P$ have the same order of zeros. If $\dim \ker Q(x, \xi) = l$, then $P(x, \xi) = 0$, so $\det P$ has a zero of order at least l at (x, ξ). Because the same holds for $\det Q$, we have proved:

Lemma 2. *If* $\dim\ker Q(x,\,\xi) = l$ *and* $Q(x,\,\xi)$ *is invertible modulo its kernel, then* $\det Q$ *has a zero of order at least* l *at* $(x,\,\xi)$*.*

3. The genericity assumptions

In the sequel, we will concentrate on a conic neighborhhood of a point in the set

$$\Sigma := \{(x,\,\xi) \in N \mid \dim\ker Q(x,\,\xi) > 1\}.$$

Note that Σ is a closed, conic subset of N. In this section, we will investigate the geometry of the symbol around Σ, with the purpose of establishing the genericity assumptions under which the normal form can be obtained. We begin with the consequences of Lemma 1 for operators with real symmetric principal symbols.

Corollary 3. *Assume that, possibly after multiplying* $\mathbf{Q}$ *with elliptic factors, the principal symbol* $Q(x,\,\xi)$ *is a real symmetric* $k \times k$*-matrix, and that*

$$\dim\ker Q(x^0,\,\xi^0) = 2.$$

Then one can split off a 2×2 *operator* $\mathbf{P}$*, with real symmetric principal symbol* $P(x,\xi)$*. Near* $(x^0,\,\xi^0)$*, we have*

$$(x,\,\xi) \in \Sigma \Longleftrightarrow P(x,\,\xi) = 0 \Longleftrightarrow \dim\ker Q(x,\,\xi) = 2,$$

and $\det Q$ *has a zero of multiplicity more than one at each point of* Σ*.*

Moreover, the rank of the Hessian of $\det Q$ *at* $(x^0,\,\xi^0)$ *is at most equal to three. If it is equal to three, then there is a conic open neighborhood* U *of* $(x^0,\,\xi^0)$ *in* $T^*M \setminus 0$*, such that* $\Sigma \cap U$ *is a smooth submanifold of codimension three in* U*.*

Proof. The symmetry implies that the range R of $Q := Q(x^0,\,\xi^0)$ is equal to the orthogonal complement of $K = \ker Q$. and $Q|_R$ is invertible from R to itself. So, with an orthonormal basis for which K is spanned by the first two basis vectors and R by the remaining $k-2$, we get the situation of Lemma 1 with $l = 2$ and $Q(x,\,\xi)$ symmetric. This proves the first part of the Corollary.

For the last statement, we observe that we can write the symmetric matrix $P(x,\,\xi)$ in the form

$$P = P(x,\,\xi) = \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix}, \tag{3.1}$$

for uniquely determined smooth functions q, r, s of $(x,\,\xi)$, homogeneous of degree m in ξ. The set Σ is equal to the intersectin of the sets where q, r and s vanish. The scalar symbol equals

$$p(x,\xi) := \det P(x,\,\xi) = q^2 - r^2 - s^2. \tag{3.2}$$

from which we see that the Hessian of p at each point of Σ has rank at most three, with equality if and only if dq, dr and ds are linearly independent. In turn, this implies that Σ is a smooth codimension 3 submanifold of $T^*M \setminus 0$. The last statement follows because the Hessian of $\det Q$ is an nonzero multiple of the Hessian of p at each point of Σ. $\square$

Our next ingredient is the canonical symplectic form

$$\sigma := \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

in T^*M. Each smooth function f in T^*M defines a Hamiltonian vector field

$$H_f = \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j},$$

or, equivalently, in a coordinate independent definition:

$$i_{H_f}\sigma = -df.$$

Because σ is closed, it follows from the homotopy formula that the Lie derivative of σ with respect to any Hamiltonian vector field is equal to zero or, equivalently, that σ is invariant under the flow of H_f.

If g is another smooth function, then we shall also use the *Poisson brackets*

$$\{f, g\} := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} = -H_f g,$$

which define a Lie algebra structure on $C^\infty(M)$.

We now turn to the Hamiltonian vector field of p, the solution curves of which on $N \setminus \Sigma$ are called the *bicharacteristic curves.* ¿From (3.2) we see that

$$H_p = 2(qH_q - rH_r - sH_s),$$

vanishes at $(x, \xi) \in \Sigma$, so it has an invariantly defined linearization

$$L = 2(dq \otimes H_q - dr \otimes H_r - ds \otimes H_s)$$

at (x, ξ), which is a linear endomorphism of the tangent space $\mathcal{T} := T_{(x,\xi)}(T^*M)$. Alternatively L can be regarded as the Hamiltonian vector field on $\mathcal{T}$ of the quadratic part of the Taylor expansion of p at (x, ξ). The rank of L is equal to the rank of the Hessian of p at (x, ξ). It is at most equal to three, with equality if and only if H_q, H_r and H_s are linearly independent at (x, ξ). We will assume this in the sequel.

Notice that the image $R := \operatorname{im} L$ is spanned by H_q, H_r and H_s at (x, ξ). On the other hand, $K := \ker L$ is equal to the intersection of the kernels of dq, dr and ds at (x, ξ), so equal to the tangent space to Σ at (x, ξ). Now the fact that the flow of H_p leaves σ invariant implies that L is an infinitesimally symplectic transformation, that is,

$$\sigma(Lu, v) + \sigma(u, Lv) = 0, \quad u, v \in \mathcal{T}.$$

It follows that the range of L is equal to the symplectic orthogonal complement of the kernel of L. Now Σ is said to be *involutive* at (x, ξ) if the symplectic orthogonal complement of $T_{(x,\xi)}\Sigma$ is contained in $T_{(x,\xi)}\Sigma$. In our situation this would mean that $R \subset K$, or $L^2 = 0$, so that the assumption that L is not nilpotent implies that Σ is not involutive at (x, ξ).

If we restrict σ to R, then its kernel is equal to $R \cap K$ and σ induces a nondegenerate antisymmetric bilinear form on $R/(R \cap K)$. Its dimension is even and at most three, so we either have $R = R \cap K$, corresponding to the case that Σ is involutive at (x, ξ), or $\dim R/(R \cap K) = 2$, in which case σ is a nonzero 2-form on $R/(R \cap K)$. The flow of L preserves the form, hence L is traceless on $R/(R \cap K)$. Therefore, either L has two purely imaginary opposite eigenvalues or two real opposite eigenvalues.

Note that $\dim(R \cap K) = 1$, so there is always a zero eigenvalue for $L|_R$. which actually leads to a nilpotent part of L in $\mathcal{T}$. For instance in the case that L is an infinitesimal rotation, this causes the bicharacteristic curves near Σ to be narrowly winding helices along the curves which are tangent to $R \cap K$.

On the basis in R of the vectors H_q, H_r and H_s at (x, ξ), the matrix of L is equal to

$$\begin{pmatrix} 0 & -2\{r, q\} & -2\{s, q\} \\ 2\{q, r\} & 0 & 2\{s, r\} \\ 2\{q, s\} & 2\{r, s\} & 0 \end{pmatrix}. \tag{3.3}$$

By computing the characteristic polynomial one recovers that it has a zero eigenvalue and that the two other eigenvalues λ satisfy

$$\lambda^2 = 4(\{r, q\}^2 + \{s, q\}^2 - \{r, s\}^2) \tag{3.4}$$

Hence, the condition that L is not nilpotent is equivalent to the condition that the right hand side of (3.4) is nonzero. If it is negative, then L defines an infinitesimal rotation in $R/(R \cap K)$, whereas L induces a hyperbolic area preserving flow in $R/(R \cap K)$ if the right hand side of (3.4) is positive.

Because $\det Q$ is a nonzero multiple of $\det P$, we get that the linearization $L_Q(x, \xi)$ of the Hamiltonian flow of $\det Q$ at $(x, \xi) \in \Sigma$ is a nonzero multiple of L, so all the conditions can be formulated in terms of Q, and are invariant under multiplication by elliptic factors and splitting off elliptic summands. We are now ready to formulate the assumptions under which we will derive our normal form. These coincide with the assumptions made by Ivrii [11], [12, Thm. 3.1 and 3.4], in his investigation of propagation of singularities and by Arnol'd [3, Sec. 8.1-8.4], [1], [2] in his study of the normal form of the characteristic set N near Σ.

Assumption 4. (1) **Q** *has a real and symmetric principal symbol* $Q(x, \xi)$.

(2) $\dim\ker Q(x^0,\,\xi^0) = 2$. *This implies that* $\det Q$ *has a zero of multiplicity at least two at* $(x^0,\,\xi^0)$, *which makes that the linearization* $L = L_Q$ *of the Hamiltonian vector field of* $\det Q$ *at* $(x^0,\,\xi^0)$ *is invariantly defined.*

(3) *The rank of* L *is equal to three and* L *is not nilpotent.*

(4) *The direction of the cone axis, given by the Euler vector field*

$$E = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j},$$

is not contained in the range of L.

(5) $n = \dim M \geq 3$.

It is clear from the above description, that the assumptions are of a generic nature. More precisely a generic, real symmetric pseudo differential principal matrix symbol will meet the variety of symmetric matrices with corank 2 transversely. The same transversality holds for differential operators, c.f. Arnold [3], Khesin [15].

4. Formal Normal Forms

Assumption 4, combined with Lemma 1, enables one to split off an elliptic summand and reduce to a two by two system with principal symbol as in (3.1). In this section we will determine the normal forms for these 2×2 systems. We will say that a smooth function R is *flat at* Σ if the full Taylor expansion of R vanishes at each point of Σ. The main theorem of the paper is

Theorem 5. *Let* $\mathbf{P}$ *be* 2×2 *system with a symmetric symbol which satisfies Assumption* 4 *at a point* $(x^0, \xi^0) \in T * M$ *of multiplicity two. There is a smooth canonical transformation* f, *homogeneous of degree one with respect to the* ξ*-variables, from a conic open neighborhood* V *of* $(0,\, dx_3) \in T^*\mathbb{R}^n$ *to a conic open neighborhood* U *of* $(x^0,\,\xi^0)$, *and a smooth mapping*

$$A : U \to GL(2, \mathbb{R}),$$

homogeneous of degree $(1 - m)/2$, *such that*

$$(A\,P\,A^t)(f(x,\,\xi)) = \begin{pmatrix} \xi_1 + \xi_2 & x_2\xi_3 \\ x_2\xi_3 & \pm(\xi_1 - \xi_2) \end{pmatrix} + R(x,\,\xi), \tag{4.1}$$

where the remainder term R *is flat at*

$$f^{-1}(\Sigma) = \{(x,\,\xi) \in V \mid \xi_1 = \xi_2 = x_2 = 0\}.$$

The plus sign in (4.1) *corresponds to the case that the nonzero eigenvalues* λ *of* L, *cf. Assumption* 4 *and* (3.4), *are purely imaginary, whereas the minus sign occurs if these eigenvalues are real.*

The main step in achieving the normal form is:

Proposition 6. *There is a $GL(2,\mathbb{R})$-valued function A defined in a conical neighbourhood U of $(x_0, \xi_0) \in \Sigma$, homogeneous of degree $1/4 - m/2$, such that APA^t equals*

$$\begin{pmatrix} \tilde{q}+\tilde{r} & \tilde{s} \\ \tilde{s} & \tilde{q}-\tilde{r} \end{pmatrix}$$

with:

$$\begin{array}{lll} \{\tilde{q},\tilde{r}\} = \{\tilde{q},\tilde{s}\} = 0 & \{\tilde{s},\tilde{r}\} = \pm 1 & \text{in the + case} \\ \{\tilde{q},\tilde{r}\} = \{\tilde{s},\tilde{r}\} = 0 & \{\tilde{s},\tilde{q}\} = 1 & \text{in the - case} \end{array} \tag{4.2}$$

where the equalities hold modulo functions which are flat at $U \cap \Sigma$.

Proof. The proof is the subject of section 5. □

Theorem 5 is now proved from Proposition 6 in the following way. In the ++ case, write

$$\xi_1 = \lambda\tilde{q},\ \xi_2 = \lambda\tilde{r},\ x_2 = \tilde{s}/\lambda,\ \xi_3 = \lambda^2,$$

in which λ is as in Proposition 7 below. By the homogeneous Darboux theorem, see e.g. Hörmander [17, Thm.21.1.9], we can locally extend these coordinates to a set of coordinates which define the desired homogeneous canonical transformation. The factor $1/\lambda$ in front of the whole matrix is eliminated if we replace A with $\lambda^{1/2}A$.

In the +− case, we switch back to the previous case by means of the homogeneous canonical transformation

$$\xi_2 \mapsto x_2\xi_3,\ x_2 \mapsto -\xi_2/\xi_3,\ \xi_3 \mapsto -\xi_3,\ x_3 \mapsto -x_3 - x_2\xi_2/\xi_3.$$

Finally, in the − case, we use

$$\xi_1 = \lambda\tilde{r},\ \xi_2 = \lambda\tilde{q},\ x_2 = \tilde{s}/\lambda,\ \xi_3 = \lambda^2.$$

Note that in general the canonical transformation does not respect the fibration

$$(x, \xi) \mapsto x : T^*\mathbb{R}^n \to \mathbb{R}^n.$$

Proposition 7. *Let $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ satisfy (4.2), and assume they are homogeneous of degree $\frac{1}{2}$. There is a positive smooth function λ, homogeneous of degree $\frac{1}{2}$, such that $\{\lambda, \tilde{q}\}$, $\{\lambda, \tilde{r}\}$, and $\{\lambda, \tilde{s}\}$ are flat at Σ.*

Proof. (Compare Roels & Weinstein [23].) The Hamiltonian vectorfields $H_{\tilde{q}}$, $H_{\tilde{r}}$ and $H_{\tilde{s}}$ commute modulo flat terms. This is based on the Jacobi identity for Poisson brackets, which is equivalent to

$$[H_f, H_g] = H_{\{g,f\}}.$$

By adding flat terms to $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ we can find $\hat{q}$, $\hat{r}$ and $\hat{s}$, such that $H_{\hat{q}}$, $H_{\hat{r}}$ and $H_{\hat{s}}$ commute. For instance, in the + case, we may take $\hat{q} = \tilde{q}$ and take $\hat{r}$ equal to the solution of $\{\hat{q}, \hat{r}\} = 0$, with $\hat{r} = \tilde{r}$ on a conic codimension one submanifold, which is transversal to $H_{\hat{q}}$. Because $H_{\hat{q}}$ is tangent to Σ, $\hat{r} - \tilde{r}$ is flat at Σ. Next let S be a conic codimension two submanifold of T^*M, transversal to $H_{\hat{q}}$ and $H_{\hat{r}}$, such that $S \cap \Sigma$ is a codimension one submanifold of Σ. The desired $\hat{s}$ is obtained by taking $\{\hat{q}, \hat{s}\} = 0$, $\{\hat{s}, \hat{r}\} = \pm 1$ and $\hat{s} = \tilde{s}$ on Σ.

The Euler vectorfield E in Assumption 4 satisfies for any functions f, g:

$$E\{f, g\} = \{Ef, g\} + \{f, Eg\} - \{f, g\}. \tag{4.3}$$

Let M_0 be a codimension 4 submanifold of T^*M through $(x^0, \xi^0) \in \Sigma$, transverse to $H_{\hat{q}}$, $H_{\hat{r}}$, $H_{\hat{s}}$, E. We choose λ equal to one on M_0 and extend it to a function homogeneous of degree $\frac{1}{2}$ on $M_1 = \mathbb{R}_{>0} \cdot M_0$, using the multiplicative action $(x, \xi) \mapsto (x, \tau\xi)$, $\tau > 0$. The requirement that λ Poisson-commutes with $\hat{q}$, $\hat{r}$, $\hat{s}$ defines it on an open subset of T^*M. From the above relation one deduces that for $f = \hat{q}, \hat{r}, \hat{s}$:

$$\{f, E\lambda - \frac{1}{2}\lambda\} = 0$$

using the homogeneity of f. Thus λ is homogeneous of degree $\frac{1}{2}$ everywhere. $\square$

5. Proof of Proposition 6

The formal normal forms will be achieved by induction on the order. The first proposition will establish the vanishing of the relevant Poisson bracket up to first order. Before proceeding with the induction step we will introduce the usual machinery associated with filtrations by orders.

In this section we have use for the function

$$e = q\{r, s\} + r\{s, q\} + s\{q, r\}.$$

At every point of Σ the Hamilton vectorfield of e spans the nullspace of the restriction to Σ of the symplectic form. This subspace is equal to the intersection of the kernel and range of L, c.f. (3.3). Recall that we wrote our symmetric symbol matrix as

$$P = \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix}.$$

With $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an invertible matrix-valued function homogeneous of degree 0, we can write

$$APA^t = \begin{pmatrix} \tilde{q}+\tilde{r} & \tilde{s} \\ \tilde{s} & \tilde{q}-\tilde{r} \end{pmatrix}$$

where

$$\begin{aligned} \tilde{q} &= \tfrac{1}{2}(a^2+b^2+c^2+d^2)q &+& \tfrac{1}{2}(a^2-b^2+c^2-d^2)r &+& (ab+cd)s \\ \tilde{r} &= \tfrac{1}{2}(a^2+b^2-c^2-d^2)q &+& \tfrac{1}{2}(a^2-b^2-c^2+d^2)r &+& (ab-cd)s \\ \tilde{s} &= (ac+bd)q &+& (ac-bd)r &+& (ad+bc)s \end{aligned} \tag{5.1}$$

Lemma 8. *We can find A and a conical neighbourhood of (x, ξ) such that on $\Sigma \cap U$*

$$(5.2) \qquad \begin{array}{lll} \{\tilde{q}, \tilde{r}\} = \{\tilde{q}, \tilde{s}\} = 0 & \{\tilde{s}, \tilde{r}\} = \pm 1 & \textit{in the + cases} \\ \{\tilde{q}, \tilde{r}\} = \{\tilde{s}, \tilde{r}\} = 0 & \{\tilde{s}, \tilde{q}\} = 1 & \textit{in the - case} \end{array}$$

Proof. We first deal with the +-case and then indicate what modifications one makes for the minus case. The equations $\{\tilde{q}, \tilde{r}\} = \{\tilde{q}, \tilde{s}\} = 0$ on Σ will be satisfied if $\tilde{q}$ is a constant multiple of the function e. This is the case if

$$(5.3) \qquad \begin{array}{c} \frac{1}{2}(a^2 + b^2 + c^2 + d^2) = \mu\{r, s\} \\ \frac{1}{2}(a^2 - b^2 + c^2 - d^2) = \mu\{s, q\} \\ (ab + cd) = \mu\{q, r\} \end{array}$$

for a constant μ. To see that this equation can be solved observe that the map:

$$A \to A^t A = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

exhibits a locally trivial fibration $GL(2, \mathbb{R}) \to O(2) \setminus GL(2, \mathbb{R})$. Its image consists of the symmetric matrices with 2 positive eigenvalues, a simply connected component of $\mathbb{R}^3$ minus a cone. A choice of sign for μ ensures that a solution can be found.

Assuming that this has been done, i.e. that $\{q, r\} = \{q, s\} = 0$ on Σ, let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ to get:

$$\tilde{q} = a^2 q \qquad \tilde{r} = a^2 r \qquad \tilde{s} = a^2 s.$$

This gives $\{\tilde{q}, \tilde{r}\} = \{\tilde{q}, \tilde{s}\} = 0$ and $\{\tilde{r}, \tilde{s}\} = a^4\{r, s\}$ on Σ we have that (5.3) is fully met.

The −-case is done similarly, swapping the role of q and r by using the fibration:

$$A \to A^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A.$$

Here we get all invertible symmetric matrices with negative determinant and we can solve an equation similar to the one above without choosing a sign. $\square$

We will use the following spaces of formal functions. Observe that q, r, s are coordinates transverse to Σ. Define the ring:

$$\mathcal{R} = C^\infty(\Sigma) \otimes_{\mathbb{R}} \mathbb{R}[[q, r, s]],$$

where $\mathbb{R}[[\ ]]$ denotes formal power series. Let $\mathcal{R}^\mu$ be the functions homogeneous of degree μ with respect to the conic structure in T^*M. All the $\mathcal{R}^\mu$ are filtered by the degree of the lowest order polynomial term in the Taylor expansion in q, r, s. We denote this filtration by

$$\mathcal{R}^\mu = \mathcal{R}_0^\mu \supset \mathcal{R}_1^\mu \supset \ldots.$$

We regard P as a symmetric matrix with coefficients in $\mathcal{R}_1^{\frac{1}{2}}$ and write $P \in S^2(\mathcal{R}_1^{\frac{1}{2}})$. Again we will write the proof out in the ++ case, the other cases are similar. Our equations can be written as the condition that

$$(5.4) \quad E_+ : \begin{pmatrix} q+r & s \\ s & q-r \end{pmatrix} \to (\{q,r\},\{q,s\},\{r,s\}-1) : S^2(\mathcal{R}_1^{\frac{1}{2}}) \to \mathcal{R}^0 \otimes \mathbb{R}^3$$

maps to zero. Above we saw that for suitable $A \in GL(2,\mathcal{R}^0)$ we have

$$E_+(APA^t) \in \mathcal{R}_1^0 \otimes \mathbb{R}^3.$$

Proposition 9. *Assume that $E_+(P) \in \mathcal{R}_k^0 \otimes \mathbb{R}^3$ for some $k > 0$. Then there is a $\delta A \in gl(2,\mathcal{R}_k^0)$ such that for $A = I + \delta A$ we have*

$$E_+(APA^t) \in \mathcal{R}_{k+1}^0 \otimes \mathbb{R}^3.$$

Proof. This proof is computational. Let $\delta A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a, b, c, $d \in \mathcal{R}_k^0$, $k > 0$. From (5.1) we see:

$$(5.5) \quad \begin{array}{rrrrl} \tilde{q} = & (1+a+d)q+ & (a-d)r+ & (b+c)s & \bmod \mathcal{R}_{2k+1} \\ \tilde{r} = & (a-d)q+ & (1+a+d)r+ & (b-c)s & \bmod \mathcal{R}_{2k+1} \\ \tilde{s} = & (b+c)q+ & (c-b)r+ & (1+a+d)s & \bmod \mathcal{R}_{2k+1} \end{array}$$

The next step is to work out the Poisson brackets appearing in (5.4). The Poisson brackets map $\mathcal{R}_k \otimes \mathcal{R}_m$ to $\mathcal{R}_{m+k-2}$, but taking the Poisson brackets with special functions one can do better. In particular,

$$(5.6) \quad f \mapsto \{q, f\} \qquad f \mapsto \{r, f\} - \frac{\partial f}{\partial s} \qquad f \mapsto \{s, f\} + \frac{\partial f}{\partial r}.$$

all map $\mathcal{R}_m$ to $\mathcal{R}_m$.

Combining the previous two sets of equations we get:

(5.7)

$$\begin{array}{rcl} \{\tilde{q},\tilde{r}\} & = & \{q,r\} + \{a+d,r\}q + \{a-d,r\}r + \{b+c,r\}s - (b+c) \bmod \mathcal{R}_{k+1} \\ \{\tilde{q},\tilde{s}\} & = & \{q,s\} + \{a+d,s\}q + \{a-d,s\}r + \{b+c,s\}s + (a-d) \bmod \mathcal{R}_{k+1} \\ \{\tilde{r},\tilde{s}\} - 1 & = & \{r,s\} - 1 + \{a-d,s\}q + \{r,b+c\}q + \{b-c,r\}r + \{b-c,s\}s \\ & & +\{a+d,s\}r + \{r,a+d\}s + 2(a+d) \bmod \mathcal{R}_{k+1}. \end{array}$$

Modulo $\mathcal{R}_{k+1}$ the functions $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ depend only on the image of δA in $\mathcal{R}_k/\mathcal{R}_{k+1}$.

We will now define suitable vector spaces and operators to discuss the solubility of these equations. We use the associated graded vector spacesmay view $\mathcal{G}R_k = \mathcal{R}_k/\mathcal{R}_{k+1}$, which is isomorphic to the space of homogeneous polynomials in q, r, s of degree k with coefficients in $C^\infty(\Sigma)$. First we solve the first two of (5.7) for

$$\alpha \equiv a - d \text{ and } \beta \equiv b + c$$

as follows. Write the equations as $\Phi_j(\alpha, \beta) = \phi_j$, $j = 1, 2$, where

$$\Phi_1(\alpha, \beta) := \beta - \{\alpha, r\}r - \{\beta, r\}s = \beta + r\partial\alpha/\partial s + s\partial\beta/\partial s,$$
$$\Phi_2(\alpha, \beta) := \alpha + \{\alpha, s,\}r + \{\beta, s\}s = \alpha + r\partial\alpha/\partial r + s\partial\beta/\partial r,$$

and

$$\phi_1 := \{q, r\} + \{a + d, r\}q,$$
$$\phi_2 := -\{q, s\} - \{a + d, s\}q).$$

In the description of Φ_1 we have used (5.6).

These operators on $\mathcal{G}R_k$ leave the $l+1$-dimensional subspace $\mathcal{G}R_{k,l}$, spanned by the $q^{k-l}r^u s^v$ with $u + v = l$, invariant. The kernel of

$$\Phi = (\Phi_1, \Phi_2) : (\mathcal{G}R_{k,l})^2 \to (\mathcal{G}R_{k,l})^2$$

is spanned by the elements $(r^{l-\mu}s^{\mu}, -r^{l-\mu+1}s^{\mu-1})$, $\mu = 1, \ldots, l$, hence $\dim\ker\Phi = l$. On the other hand, its image is contained in the kernel of the mapping

$$\Psi : (\mathcal{G}R_{k,l})^2 \to \mathcal{G}R_{k,l-1} : (\gamma, \delta) \mapsto \{\gamma, s\} + \{\delta, r\} = \partial\gamma/\partial r - \partial\delta/\partial s.$$

The codimension of $\ker\Psi$ is equal to l, so we get that the range of Φ is equal to the kernel of Ψ.

On the other hand, using the Jacobi identity and the fact that $\{q, r\}$, $\{q, s\}$ and $\{r, s\} - 1$ all belong to $\mathcal{R}_k$, we get that $\Psi(\phi_1, \phi_2) = 0$. The conclusion is that the equations $\Phi_j(\alpha, \beta) = \phi_j$ can indeed be solved.

Finally we solve the third equation of (5.7) for $\gamma \equiv a + d \in \mathcal{G}R_k$, by substituting the solution for $b + c$ and $a - d$ found above in terms of $a + d$. For $b - c$ we can substitute anything. The relevant operator is now:

$$\Xi : \mathcal{G}R_k \to \mathcal{G}R_k : \gamma \mapsto 2\gamma + \{r, \gamma\}s - \{s, \gamma\}r = 2\gamma + s\partial\gamma/\partial s + r\partial\gamma/\partial r.$$

This map is linear over $C^\infty(\Sigma)$ and $\Xi(q^\rho r^\sigma s^\tau) = (2 + \sigma + \tau)q^\rho r^\sigma s^\tau$. So Ξ is invertible, which allows for unique solubility. $\square$

To prove Proposition 6 we call on Borel's lemma [4], [17] which supplies a smooth matrix function which has the Taylor expansion prescribed by the lemma above.

6. Examples from mathematical physics

We first take a look at the Maxwell equations and show that they can satisfy Assumption 4, with either sign for the sum of the squares of the eigenvalues of L, cf. Ivrii [11]. The Maxwell system in a dielectric medium is a system for $\mathbb{R}^3$ valued functions E_j on $\mathbb{R}^3 \times \mathbb{R}$, which can be written as:

$$\sum_j \epsilon_{ij}(x)\frac{\partial^2 E_j}{\partial t^2} = (\nabla \times \nabla \times E)_i.$$

where $\epsilon_{ij}(x)$ is the dielectric tensor, which is dependent of the space variables and which forms a symmetric 3 by 3 matrix with 3 positive eigenvalues. Assume that ϵ is diagonal, then we can write the symbol matrix as:

$$(x,\xi) \to \begin{pmatrix} \epsilon_1(x)\xi_0^2 - \xi_2^2 - \xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \epsilon_2(x)\xi_0^2 - \xi_1^2 - \xi_3^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \epsilon_3(x)\xi_0^2 - \xi_1^2 - \xi_2^2 \end{pmatrix}.$$

The determinant of the symbol can be expressed as:

$$p = \xi_0^2(bc - \xi_2^2\alpha),$$

with

$$\begin{aligned} b &= \epsilon_2\xi_0^2 - \xi_1^2 - \xi_3^2, \\ c &= \epsilon_1\epsilon_3\xi_0^2 - \epsilon_1\xi_1^2 - \epsilon_3\xi_3^2, \\ \alpha &= (\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_2)\xi_0^2 + \epsilon_2(b - \xi_2^2) + c. \end{aligned}$$

As is customary in optics, we assume that $\xi_0 \neq 0$, and we also assume that the eigenvalues of the dielectric tensor satisfy that $\epsilon_1(x) < \epsilon_2(x) < \epsilon_3(x)$.

If the gradient $\partial p/\partial\xi$ of p with respect to the ξ-variables is equal to zero, then $p = \frac{1}{2}\sum \xi_j \partial p/\partial \xi_j = 0$, $\xi_2 = 0$, $b = 0$, $c = 0$. Conversely, if $\xi_2 = b = c = 0$, then the rank of the symbol matrix is equal to one, as is easily verified. The conclusion is that, away from $\xi_0 = 0$, Σ corresponds to $\xi_2 = b = c = 0$, and that outside Σ we even have $\partial p/\partial\xi \neq 0$. Also note that Σ is parametrized by:

$$\xi_2 = 0, \quad \xi_1^2 = \xi_0^2\,\frac{\epsilon_3(\epsilon_2 - \epsilon_1)}{\epsilon_3 - \epsilon_1}, \quad \xi_3^2 = \xi_0^2\,\frac{\epsilon_1(\epsilon_3 - \epsilon_2)}{\epsilon_3 - \epsilon_1},$$

with $\xi_0 \neq 0$. (Cf. Kline and Kay [16] and Landau & Lifshitz [20].)

Using that

$$\alpha = (\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_2)\xi_0^2 > 0 \quad \text{at} \quad \Sigma,$$

we get that the set U where $\alpha > 0$ is a conic open neighborhood of Σ. In U we can write $p = \xi_0^2(bc - a^2)$, with $a := \xi_2\sqrt{\alpha}$. Also, Σ is determined by the equations $a = b = c = 0$ in U.

Because da, db and dc are linearly independent, we see that the rank of D^2p is equal to three. Substituting $\xi_0 b = q + r$, $\xi_0 c = q - r$, $\xi_0 a = s$, we have (3.2). Using (3.4) and scaling at $\xi_0^2 = 1$, the equation for the eigenvalues λ of DH_p reads

$$\lambda\,(\lambda^2 - \{b,\, c\}^2 + 4\{c,\, a\}\,\{a,\, b\}) = 0,$$

or equivalently

$$\lambda\,(\lambda^2 - \{b,\, c\}^2 + 4\{c,\, \xi_2\}\,\{\xi_2,\, b\}(\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_2)) = 0$$

Because $\{f, \xi_2\} = \partial f/\partial x_2$ and $\{b, c\}$ only involves partial derivatives of the ϵ_j with respect to x_1 and x_3, we get quite simple expressions if the ϵ_j only depend on x_2. In this case the sum of the squares of the eigenvalues is equal to a positive multiple of

$$\epsilon_2'[\epsilon_3(\epsilon_3 - \epsilon_2)\epsilon_1' + \epsilon_1(\epsilon_2 - \epsilon_1)\epsilon_3'],$$

in which the prime denotes differentiation with respect to x_2. Clearly both signs can occur.

Next we turn to the equations for elastodynamic waves. These are again a system for an $\mathbb{R}^3$ valued function on $\mathbb{R}^4$. Let $\rho(x)$ be the density of the material, $c_{pqrs}(x)$ the moduli of elasticity and $u_i(x)$ the displacement vector. The equations of motion are:

$$\rho\, \partial_t u_p - c_{pqrs}\partial_q\partial_s u_r = 0. \tag{6.1}$$

The elasticity constants satisfy pointwise the symmetries: $c_{pqrs} = c_{qprs} = c_{rspq}$ and the positivity $c_{pqrs}a_{pq}a_{rs} \geq 0$ for any symmetric matrix a_{pq}.

Here we shall show that both signs are again realized in this system. Moreover, we observe once more that the derivatives of the material properties are responsible for the sign.

It takes considerable care to determine the conical points of the characteristic variety of the system (6.1). It is known that in the projectivized cotangent bundle the number of conical points is always even, must lie between 0 and 16 and can take any value satisfying these constraints, see Holm [10].

In a medium of cubic symmetry there are many additional relations between the c_{pqrs} and one can exploit these to compute the sign of the system at the points of multiplicity. In particular we can call on the results of Burridge [6] who computes the Hessian of the determinant of the symbol at these points. Working in a single fibre and putting $\xi_0 = 1$, he finds a point of multiplicity 2 with coordinates $(1, \alpha, \alpha, \alpha)$, for a certain number α. At this point he introduces an orthogonal set of axis, the first with direction $(0, 1, 1, 1)$ and all in the hyperplane $\xi_0 = 1$. In the linear coordinates of this system of axes he gets:

$$p(x, \eta) = R\zeta_1^2 - S(\zeta_2^2 + \zeta_3^2)$$

where R, S are positive functions of the moduli of elasticty which can vary independently. From this we immediately deduce that both signs can occur.

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Convolutions of generalized functions, applied to diffraction theory of crystals and quasicrystals.

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EXTENDED ABSTRACT.

X-ray diffraction of crystals (and quasicrystals) can be described by Fourier theory and convolution theory for generalized functions. The idea is that a crystal is a generalization of the "Poisson comb": a countable sum of delta functions in space, with the property that its Fourier transform has again such a discrete structure. One of course needs a theory of generalized functions (distributions) for it. Because of the symmetry in this particular problem it seems to be atractive to take a distribution theory with Fourier invariance properties. One of the classes introduced by Gelfand-Schilow is an excellent candidate for it. It is the class that was represented in a different form by J. Korevaar's "Hermite pansions" [7] and by a particular kind of traces based on a semigroup of operators in a space of smooth functions in [1]. Each one of these different introductions to one and the same space of generalized functions has its own merits.

Gelfand-Schilow's introduction of the class is in the usual style of linear functionals on a space of test functions. For the class to be considered here, the test function space has to be the class $S_{1/2}^{1/2}$ (see [4], Ch. 4, section 2.3). This class will be called S from here on. It is the set of all entire functions f for which there are positive constants A, B and M such that

$$|f(x+iy)| \leq M \exp(-Ax^2 + By^2)$$

for all real x and y.

Korevaar works on a different principle. Any square integrable function on the real line can be expanded as a series of Hermite functions, and can therefore be characterized by means of the sequence of coefficients $(c_1, c_2, \ldots)$ in that expansion. For the square integrable functions one gets all sequences with $|c_1|^2 + |c_2|^2 + \cdots < \infty$. From then on, Korevaar uses the sequences instead of the functions, and extends the class by liberalizing the condition to $c_n = O(e^{\epsilon n})$ (for all $\varepsilon > 0$).

In the approach of [1] the same space of smooth functions is used as by Gelfand-Schilow, but the path from the set S of smooth functions to the set S^* of generalized functions is entirely different. It gives a prominent role to a particular semigroup consisting of integral operators N_α on S, defined for $\alpha > 0$, with $N_\alpha N_\beta = N_{\alpha+\beta}$ for all $\alpha > 0$, $\beta > 0$. These operators, called *smoothing operators*, commute with many other important operators, like the Fourier transform.

The class S^* of all generalized functions is now introduced as the set of all mappings of the set of positive numbers into S with the property that $N_\alpha F(\beta) = F(\alpha + \beta)$ for all $\alpha > 0$, $\beta > 0$. Such mappings can be called *traces*. Particular traces are those

which are generated by elements of S: if $f \in S$ then the mapping that sends α to $N_\alpha f$ is a trace. Identifiying f with this trace we have the natural embedding of S into S^*.

The advantage of this approach is that one can conceive all sorts of operators and structures in S (where life is easy) and extend them to S^* by simple algebraic operations.

In [2] and [3] this approach to generalized functions was used for the Fourier theory of quasicrystals.

Among crystallographers it is taken for granted that X-ray diffraction patterns of crystals (or quasicrystals) are related to Fourier theory in the following way. A narrow X-ray beam passes through a crystal and gives a diffraction pattern on a photographic plate. The crystal can be considered as a distribution in space. Now the idea is that the picture on the plate can be described as the intersection of the Fourier transform of that distribution with a plane through the origin (the direction of that plane is determined by the direction of the beam).

In order to derive such a statement it is attractive to have an infinite crystal, an infinitely small beam, a photographic plate at infinity, and similar simplifications that seem to be quite far from reality. Until now, a satisfactory treatment of all this has not been given in the literature.

Treating the effect of the finiteness of beam and crystal in the framework of Fourier theory and generalized functions, a theory of convolutions of elements of S^* is required. A very satisfactory theory meeting these requirements was given by A.J.J.M. Janssen in [6]. Some of the material needed for it is contained in [5].

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Logarithmic convexity of L^2–norms for solutions of linear elliptic equations

Raymond Brummelhuis*

Dedicated to prof. Jaap Korevaar on the occasion of his 70th birthday

In the course of their work on equilibrium and minimal field distributions of finitely many electrons on the sphere in $\mathbf{R}^3$, J. Korevaar and J. Meyers discovered the following surprising analogue for harmonic functions of the classical Hadamard 3–circles theorem:

Theorem 1. *(Korevaar and Meyers, cf. [3], [4]): Let $0 < \rho < r < R < \infty$. There exists a constant $\alpha \in (0,1)$, only depending on $\rho/R, r/R$ and n such that for all u harmonic on the ball $B(0,R) \subseteq \mathbf{R}^n$,*

$$\|u\|_{\infty,r} \leq \|u\|_{\infty,\rho}^{\alpha}\|u\|_{\infty,R}^{1-\alpha} \tag{1}$$

Here, and in the sequel, $\|\cdot\|_{p,r}$ $(1 \leq p \leq \infty)$ denotes the L^p–norm on the r–sphere $S(0,r)$, with respect to the rotation–invariant measure normalized to 1.

The classical Hadamard theorem, which is (1) for *holomorphic* u on an annulus $\{z : \rho \leq |z| \leq R\} \subseteq \mathbf{C}$, with $\alpha = \log(r/R)/\log(\rho/R)$, is proved by exploiting the subharmonicity of $\log|u(z)|$. However, for u harmonic $\log|u(z)|$ is in general far from being subharmonic. An arithmetic version of (1) for subharmonic u is of course well–known (cf. e.g. [1]).

We briefly sketch Korevaar and Meyers' elegant proof of (1). The key observation is the following L^2–version of (1):

$$\|u\|_{2,r} \leq \|u\|_{2,\rho}^{\beta}\|u\|_{2,R}^{1-\beta}, \tag{1'}$$

where, in fact, β is the exponent from the classical Hadamard theorem; (1′) easily implies the following, weaker version of (1):

$$\|u\|_{\infty,r} \leq C\|u\|_{\infty,\rho}^{\alpha}\|u\|_{\infty,R}^{1-\alpha}, \tag{1''}$$

for some constants C and α depending on $\rho/R, r/R$ and n, since the L^∞–norm of u on any sphere can be estimated in terms of the L^2–norm of u on a slightly larger sphere (e.g. by using the Poisson integral formula). Careful estimates, in combination with the usual arithmetic Hadamard inequality for subharmonic functions, show that one may in fact take $C = 1$; cf. [3] for details.

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Now to prove $(1')$, Korevaar and Meyers observed, by writing u as a sum of spherical harmonics, that $\|u\|_{2,r}^2$ is given by a convergent power series in r, with non-negative coefficients. An easy computation (or alternatively, an application of the classical Hadamard theorem for holomorphic functions) then shows that $\log \|u\|_{2,r}$ is a convex function of $\log r$.

It is natural to ask whether suitable analogues of $(1')$ and $(1'')$ also hold for solutions of a general second order homogeneous elliptic linear partial differential equation with real-valued (variable) coefficients

$$\nabla \cdot (A(x)\nabla u) + B(x) \cdot \nabla u + c(x)u = 0,$$

assuming that $c(x) \leq 0$, so that the solutions u satisfy the maximum principle. Recently the author could prove that this is indeed the case, with $S(0,r)$ the geodesic r-sphere in the Riemannian metric with metric tensor $A(x)^{-1}$. The main step is again to prove an inequality like $(1')$ (with perhaps an extra constant $C = C(\rho, r, R)$ on the right and a different β). The implication $(1') \to (1'')$ again follows from known properties of solutions of elliptic equations; one can for example use a well-known inequality of J. Moser [5] to majorize the L^∞-norm of u on a compact subset by the L^2-norm on a larger compact set.

The purpose of this note is to explain this generalization of $(1')$ in a typical "model case", in which we replace the ball by a half-space, and restrict somewhat (but not essentially) the class of elliptic equations under consideration. This allows us to present the main idea of the proof of the general case while avoiding technicalities.

We will consider formally self-adjoint 2^{nd} order real elliptic equations of the form

$$Lu = \frac{\partial^2 u}{\partial y^2} + \nabla_x \cdot (A(x,y)\nabla_x u) = 0 \tag{3}$$

on the upper half space

$$\mathbf{R}_+^{n+1} = \{(x,y) : x \in \mathbf{R}^n, y > 0\}.$$

Here $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n})$, the gradient in the x-variables, and $A = A(x,y) = (A_{jk}(x,y))_{jk}$ is a bounded Lipschitz-function on $\mathbf{R}_+^{n+1}$, with values in the real symmetric positive definite $n \times n$ matrices. We moreover assume A to be *uniformly elliptic*, that is, we assume that there exists a constant $c > 0$ such that

$$c \mid \xi \mid^2 \leq \ (A(x,y)\xi, \xi) \ , \forall \xi \in \mathbf{R}^n, \ (x,y) \in \mathbf{R}_+^{n+1}.$$

The simplest example of such an operator is of course the Laplace operator:

$$\Delta = \partial_y^2 + \sum_{j=1}^{n} \partial_{x_j}^2$$

and the natural analogue of $(1')$ in the upper half space context is:

Proposition 2. *Suppose that u is harmonic on $\mathbf{R}_+^{n+1}$ and that*

$$(4) \qquad sup_{y>0}\ \|u(y)\|_2^2 \ = \int_{\mathbf{R}^n} u(x,y)^2 dx < \infty$$

Then $\log \|u(y)\|_2^2$ *is a convex function of y.*

Proof. Harmonic functions on the upper half space satisfying a Hardy-condition (4) have a Poisson–representation as:

$$(5) \qquad u(x,y) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-y|\xi|+i<x,\xi>} \hat{f}(\xi) d\xi,$$

where $f(x) = \lim_{y\to 0} u(x,y)$ (limit in L^2–sense) and $\hat{f}$ is the Fourier–transform of f, cf. for example Stein [6].
Using (5), Plancherel and the Cauchy–Schwarz inequality, one easily proves that $\frac{d^2}{dy^2} \log \|u(y)\|_2^2 \geq 0$.

Remark. A similar result with a similar proof for harmonic functions in a strip can be found in Janson and Peetre [2]. The proof of theorem 3 below when specialized to the Laplace operator will provide another proof of this proposition, without use of the Fourier transform.

We now wish to generalize this proposition to solutions of $Lu = 0$. In the sequel we will be somewhat cavalier about the precise decay of $u(x,y)$ as $|(x,y)| \to \infty$, and just assume it to be sufficient to justify all subsequent integrations and integrations by parts. Also, we will only consider real–valued classical (C^2) solutions.

The main result of this paper is:

Theorem 3. *There exists a function $h = h(\eta) > 0$, defined for $\eta > 0$ such that for any u on $\mathbf{R}_+^{n+1}$ satisfying $Lu = 0$, $\log \|u(h(\eta))\|_2$ is a convex function of η. In fact, we can take*

$$(6) \qquad h(\eta) = \frac{c}{\gamma} \log(\frac{\gamma}{c}\eta + 1),$$

where c is the constant of ellipticity in (4), and

$$\gamma = \| \max(\partial_y A, 0)\|_\infty$$

(which is finite, since we assumed A to be Lipschitz). We are taking here the positive part in the sense of functional calculus of symmetric matrices, i.e. $\partial_y A$ composed with the projection onto its positive eigenspace. If $\gamma = 0$, we take the limit in (6): $h(\eta) = \eta$.

Remark. It is in general not true that $\log \|u(y)\|_2$ is a convex function of y: one first has to make a change of variables $y = h(\eta)$. It is of course no great surprise that a single non–convex function can be changed into a convex one by some change of variables. The point of the theorem, however, is that one can do this in one stroke, for all solutions u of $Lu = 0$ simultaneously.

Proof of theorem 3: Let

$$F(\eta) = F(\eta; u) := \int_{\mathbf{R}^n} u(x, h(\eta))^2 dx$$

We have to show that for a suitable choice of $h = h(\eta)$

$$\frac{d^2}{d\eta^2} \log F(\eta) = \frac{F''(\eta)F(\eta) - F'(\eta)^2}{F(\eta)^2} \geq 0, \tag{7}$$

for all u satisfying $Lu = 0$. To simplify notations, we will write in the sequel

$$\int_{y=h(\eta)} v \qquad \text{for} \qquad \int_{\mathbf{R}^n} v(x, h(\eta))dx$$

and

$$\int\int_{y \geq h(\eta)} v \qquad \text{for} \qquad \int_{h(\eta)}^{\infty} \int_{\mathbf{R}^n} v(x, y)dxdy.$$

Also, we will just write ∇ instead of ∇_x. With these conventions:

$$F'(\eta) = 2h'(\eta) \int_{y=h(\eta)} u\partial_y u \tag{8}$$

and

$$\begin{aligned} F''(\eta) = 2h''(\eta) \int_{y=h(\eta)} u\ \partial_y u\ &+ 2h'(\eta)^2 \int_{y=h(\eta)} (\partial_y u)^2 \\ &+ 2h'(\eta)^2 \int_{y=h(\eta)} u\ \partial_y^2 u. \end{aligned} \tag{9}$$

Since $\partial_y^2 u = -\nabla \cdot (A\nabla u)$, an integration by parts shows that the last integral in (9) equals

$$2h'(\eta)^2 \int_{y=h(\eta)} (A\nabla u, \nabla u)$$

We next use the following simple Rellich–type identity:

Lemma. For any $a > 0$,

$$\int_{y=a} (\partial_y u)^2 = \int_{y=a} (A\nabla u, \nabla u)\ +\ \int\int_{y\geq a} ((\partial_y A)\nabla u, \nabla u),$$

where $\partial_y A = (\partial_y A_{jk}(x, y))_{j,k}$.

Proof.

$$\int_{y=a} (\partial_y u)^2 = -2\int_a^\infty \int_{\mathbf{R}^n} \partial_y u \ \partial_y^2 u \ dxdy$$
$$= -2\int_a^\infty \int_{\mathbf{R}^n} (A\nabla u, \nabla(\partial_y u)) \ dxdy,$$

since $\partial_y^2 u + \nabla \cdot (A\nabla u) = 0$, and through integration by parts. Since

$$2(A\nabla u, \nabla(\partial_y u)) = \partial_y(A\nabla u, \nabla u) - ((\partial_y A)\nabla u, \nabla u),$$

the lemma follows.

If we use the lemma, (9) becomes

$$F''(\eta) = 2h''(\eta)\int_{y=h(\eta)} u \ \partial_y u + 4h'(\eta)^2 \int_{y=h(\eta)} (\partial_y u)^2 - 2h'(\eta)^2 \int\int_{y\geq h(\eta)} ((\partial_y A)\nabla u, \nabla u)$$

To have $(\log F(\eta))'' \geq 0$, we try to let the first integral compensate the last. To accomplish this, rewrite the first as a volume integral:

$$2h''(\eta)\int_{y=h(\eta)} u\partial_y u = -2h''(\eta)\int\int_{y\geq h(\eta)} (\partial_y u)^2 + u \ \partial_y^2 u$$
$$= -2h''(\eta)\int\int_{y\geq h(\eta)} (\partial_y u)^2 + (A\nabla u, \nabla u)),$$

again by the differential equation and integration by parts. Now,

$$(A\nabla u, \nabla u) \geq c|\nabla u|^2$$

and

$$((\partial_y A)\nabla u, \nabla u) \leq (\max(\partial_y A, 0)\nabla u, \nabla u) \leq \gamma \mid \nabla u \mid^2$$

Therefore, if $h''(\eta) \leq 0$,

$$2h''(\eta)\int_{y=h(\eta)} u \ \partial_y u \ - \ 2h'(\eta)^2 \int\int_{y\geq h(\eta)} ((\partial_y A)\nabla u, \nabla u)$$
$$\geq 2(-ch''(\eta) - \gamma h'(\eta)^2)\int\int_{y\geq h(\eta)} \mid \nabla u \mid^2 .$$

Hence

$$F''(\eta) \geq 4h'(\eta)^2 \int_{y=h(\eta)} (\partial_y u)^2 \tag{10}$$

if $-ch''(\eta) - \gamma h'(\eta)^2 = 0$, which is the case if $h(\eta) = \frac{c}{\gamma}\log(\frac{\gamma}{c}\eta + 1)$, and (7) then follows from (8), (10) and the Cauchy–Schwarz inequality. This proves the theorem.

Remark. The proof above uses the fact that $A(x, y)$ is a Lipschitz–function in y, uniformly in x. It would be interesting to know whether this regularity assumption on the coefficients of L can be weakened, e.g. to A continuous or even to $A \in L^\infty$ only. This would greatly enhance the potential usefullness of theorems like 3 in non–linear PDE theory.

It is natural to ask what remains true of theorem 3 for solutions of a non–homogeneous equation $L(u) = g$. One could think, for example, of L–subharmonic functions: $L(u) \geq 0$. Korevaar and Meyers already observed that (1) will in general be false for subharmonic functions: consider, for example, $u = \max(\log|z|, 0)$ on $\mathbf{C}$. The proof of theorem 3 allows us to further analyze this question. Let us agree, from the on–set, that we will only consider u's for which $\|u(y)\|_2 \neq 0$ for all y. Then careful analysis of the proof of theorem 3 shows that if $h''(\eta) = -Kh'(\eta)^2$ with $K \geq \gamma/c$, then

$$(\log F(\eta))'' \geq \frac{2h'(\eta)^2}{F(\eta)} \cdot \Big(K \int\int_{y \geq h(\eta)} (\partial_y u)^2 + 2 \int\int_{y \geq h(\eta)} (\partial_y u) Lu + K \int\int_{y \geq h(\eta)} uLu + \int_{y = h(\eta)} uL(u)\Big) \tag{11}$$

Now since $2|ab| \leq a^2 + b^2$,

$$\left|2 \int\int_{y \geq h(\eta)} (\partial_y u) Lu\right| \leq \int\int_{y \geq h(\eta)} (\partial_y u)^2 + \int\int_{y \geq h(\eta)} (Lu)^2$$

and we obtain the following corollary of (the proof of) theorem 3:

Corollary 4. *Suppose that $u \cdot Lu \geq 0$ (for example, u is a non-ne ve, L–subharmonic function) and that, moreover, u satisfies the differential inequality*

$$|Lu| \leq C \cdot |u|$$

for some $C > 0$. Let $h(\eta) = K^{-1}\log(K\eta + 1)$ with $K = \max(c^{-1}\gamma, C, 1)$. Then $\log\|u(h(\eta))\|_2$ is a convex function of $\eta > 0$.

Proof. Use that $(Lu)^2 \leq C \cdot |u| \cdot |Lu| = C \cdot u \cdot Lu$ and hence

$$-\int\int_{y \geq h(\eta)} (Lu)^2 \geq -C \int\int_{y \geq h(\eta)} uLu,$$

which can be compensated by the third integral in (11), since $u\ Lu \geq 0$.

This corollary can for example be applied to non-negative solutions of a semi–linear equation

$$Lu = F(u)$$

with non-negative and sub-linear right hand side:

$$0 \leq F(r) \leq C\,|r|, \quad r \in \mathbf{R}.$$

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On L^p-Boundedness of Affine Frame Operators

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Dedicated to Professor Jaap Korevaar on the occasion of his 70th birthday

Summary. This paper is concerned with a study of the family of affine frame operators

$$(T_\Theta f)(x) := \sum_{j,k\in \mathbb{Z}} \theta_{j,k} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x),$$

induced by some $\psi \in L^2(\mathbb{R})$, where $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$ and $\Theta = \{\theta_{j,k}\} \in \ell^\infty(\mathbb{Z}^2)$. With very mild decay assumption on ψ, T_Θ can be considered as linear operators on $L^p(\mathbb{R})$, $1 < p < \infty$, and the objective of this paper is to identify the family $\mathcal{F}$ of those ψ that give rise to bounded linear operators T_Θ on L^p. The main result is that any $\psi \in Lip\ \alpha$, $0 < \alpha < 1$, *satisfying* $O(|x|^{-1-\epsilon})$ at infinity for some $\epsilon > 0$, is in $\mathcal{F}$ provided that it has zero mean.

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1. Introduction and results

Recently, there has been much interest in the study of series representation of functions f in $L^2 := L^2(\mathbb{R})$ in terms of dilations and translations of a single function $\psi \in L^2$. The most popular consideration is an infinite series of the form

$$f(x) \sim \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(x), \tag{1.1}$$

where

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k). \tag{1.2}$$

For the representation (1.1) to be useful, the family $\{\psi_{j,k}\colon\ j,k \in \mathbb{Z}\}$ is usually required to be a *frame* (cf. [1, p. 68] and [4, p. 56]). If the representation (1.1) must also be unique for all $f \in L^2$, then this frame is necessarily an unconditional basis of L^2 (cf. [6]). In particular, if $\{\psi_{j,k}\}$ is an orthonormal (o.n.) basis, then ψ is called an ***o.n. wavelet*** (cf. [1,4,5]).

Any frame $\{\psi_{j,k}\}$ must be a ***Bessel sequence***, defined by

$$\sum_{j,k\in\mathbb{Z}} |\langle f,\psi_{j,k}\rangle|^2 \le B\|f\|_2^2, \qquad f \in L^2, \tag{1.3}$$

for some positive constant B, called a Bessel bound. That is, only the upper frame bound is used as Bessel bound in the definition of a Bessel sequence. Here and throughout, the standard notations for inner product and norm for L^2 are used. By modifying a result in [6], the following lemma provides an operator approach to the study of affine frames as discussed in our earlier work [3].

Lemma 1. *Let $\psi \in L^2$. Then the family $\{\psi_{j,k}\}$, $j,k \in \mathbb{Z}$, defined by (1.2) is a Bessel sequence with Bessel bound B as in (1.3) if and only if the linear operator*

$$(T_I f)(x) := \sum_{j,k\in\mathbb{Z}} \langle f,\psi_{j,k}\rangle\psi_{j,k}(x) \tag{1.4}$$

is bounded in L^2 with $\|T_I\| \le B$.

As an application of the above lemma, we can reformulate a result in Meyer [5, pp. 270-271] as follows.

Theorem A. *Let $\psi \in L^2$ satisfy both*

$$(1.5)\qquad \begin{cases} |\psi(x)| \le C(1+|x|)^{-1-\epsilon}; \\ |\psi(x)-\psi(y)| \le C|x-y|^{\alpha}, \qquad x,y \in \mathbf{R}, \end{cases}$$

for some $\varepsilon > 0$, $0 < \alpha \le 1$, and

$$(1.6)\qquad \int_{-\infty}^{\infty} \psi(x)dx = 0.$$

Then T_I as defined in (1.4) is a bounded linear operator in L^2.

In order to extend this study to $L^p := L^p(\mathbf{R})$, we consider the linear operators

$$(1.7)\qquad (T_\Theta f)(x) := \sum_{j,k\in\mathbf{Z}} \theta_{j,k}\langle f, \psi_{j,k}\rangle \psi_{j,k}(x),$$

where $\Theta = \{\theta_{j,k}\} \in \ell^\infty := \ell^\infty(\mathbf{Z}^2)$, which are initially defined on the class $\mathcal{D} := \mathcal{D}(\mathbf{R})$ of compactly supported C^∞ functions on $\mathbf{R}$, and then extended to all of L^p as follows.

Definition 1. *T_Θ, as defined above, is called a bounded linear operator in L^p, where $1 < p < \infty$, if*

$$\langle T_\Theta f, g\rangle := \sum_{j,k} \theta_{j,k}\langle f, \psi_{j,k}\rangle\langle \psi_{j,k}, g\rangle$$

is convergent and there exists a positive constant A_p such that

$$(1.8)\qquad \sup_{\|g\|_q\le 1} |\langle T_\Theta f, g\rangle| \le A_p\|g\|_q\|f\|_p$$

for all $f, g \in \mathcal{D}$, where $p^{-1} + q^{-1} = 1$.

Hence, (1.8) can be extended to all $f \in L^p$ and $q \in L^q$. An L^p version of Theorem A is the following (see Daubechies [4, p. 296]).

Theorem B. *Let ψ be a measurable function on $\mathbf{R}$ such that*

$$(1.9)\qquad \begin{cases} |\psi(x)| \le C(1+|x|)^{-1-\epsilon}; \\ |\psi'(x)| \le C(1+|x|)^{-1-\epsilon}, \qquad x \in \mathbf{R}, \end{cases}$$

and that

$$(1.10)\qquad \{\psi_{j,k}\colon\ j,k \in \mathbf{Z}\} \text{ is an o.n. basis of } L^2.$$

Then for any $\Theta = \{\theta_{j,k}\}$ with $\theta_{j,k} = \pm 1$, T_Θ is a bounded linear operator in L^p, $1 < p < \infty$.

The objective of this paper is to establish the following sharper result.

Theorem 1. *Let ψ be a measurable function on* $\mathbf{R}$ *satisfying (1.5). Then* T_Θ *is a bounded linear operator in* L^p *for any* $\Theta \in \ell^\infty$ *where* $1 < p < \infty$, *if and only if* ψ *satisfies (1.6).*

It is clear that (1.5) is weaker than (1.9), and it is also a wellknown fact that for ψ to be an o.n. wavelet (as defined by (1.10)), it must satisfy the zero-mean condition in (1.6) (cf. [1, p. 7] and [4, p. 7]). From the proof of Theorem 1, we will see that the operators T_Θ also satisfy the following.

(i) T_Θ is of weak type (1,1), in the sense that a positive constant C exists, with

$$|\{x: |(T_\Theta f)(x)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1, \qquad \lambda > 0, f \in L^1;$$

(ii) T_Θ is a bounded linear operator from H^1 to H^1; and

(iii) T_Θ is a bounded linear operator from BMO to BMO.

As a consequence of Theorem 1, we also have the following result.

Theorem 2. *Let* $\psi \in L^2$ *satisfy (1.5)-(1.6), such that* $\{\psi_{j,k}\}$ *is an unconditional basis of* L^2 *and that its dual basis is also generated by some* $\tilde{\psi} \in L^2$ *in the same way as (1.2), with* $\tilde{\psi}$ *satisfying (1.5)-(1.6). Then* $\{\psi_{j,k}\}$ *is also an unconditional basis of* L^p *for any p,* $1 < p < \infty$.

2. Proofs of the results

We first establish the necessary condition (1.6) in Theorem 1. For this purpose, we recall from [4, p. 63] that for ψ to satisfy (1.3), it must also satisfy

$$\text{(2.1)} \qquad \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

(and in fact, this integral cannot exceed $2B\ln 2$; see [2]). Now, by consecutive applications of the duality principle and the interpolation theorem, we see that if T_Θ is bounded in L^p, where $1 < p < \infty$, then it must also be bounded in L^2. Hence, considering $\theta_{ij} = 1$, we conclude, by applying Lemma 1, that ψ must satisfy (1.3), and hence, (2.1). Since $\psi \in L^1$, we have $\hat{\psi} \in C(\mathbf{R})$ so that $\hat{\psi}(0) = 0$. This establishes the necessary condition (1.6).

To prove that (1.6) is a sufficient condition, we only need to verify that T_Θ, where $\Theta \in \ell^\infty$, is a Calderón-Zygmund operator with kernel

$$\text{(2.2)} \qquad K(x,y) := \sum_{j,k\in\mathbf{Z}} \theta_{j,k}\psi_{j,k}(x)\overline{\psi_{j,k}(y)}.$$

Indeed, it is wellknown (cf. Meyer [5, p. 233]) that a Calderón-Zygmund operator is necessarily bounded in $L^p(\mathbf{R})$ for any p, $1 < p < \infty$. That is, to complete the proof of Theorem 1, it suffices to verify the following properties.

(a) $$|K(x,y)| \le C|x-y|^{-1},$$

(b) $$|K(x,y) - K(x',y)| \le C\frac{|x-x'|^\delta}{|x-y|^{1+\delta}},$$

(c) $$|K(y,x) - K(y,x')| \le C\frac{|x-x'|^\delta}{|x-y|^{1+\delta}},$$

for some $\delta > 0$, where $x \ne y$, $|x-x'| \le \frac{1}{2}|x-y|$, and

(d) the linear operator T_Θ is bounded in L^2.

In order to establish (a), (b) and (c), we need the following.

Lemma 2. *Let $\Phi(t)$ be a nonnegative and decreasing function on $[0,\infty)$ that satisfies*

$$\Phi(0) < \infty \quad \text{and} \quad \int_0^\infty \Phi(t)dt < \infty. \tag{2.3}$$

Then there exists a constant $C > 0$ such that

$$\sum_{k\in\mathbf{Z}} \Phi(|x-k|)\Phi(|y-k|) \le C\Phi\left(\frac{|x-y|}{2}\right), \qquad x,y \in \mathbf{R}. \tag{2.4}$$

Proof. If $|y-k| \ge \frac{1}{2}|x-y|$, then by the monotonity of Φ, we see that

$$\Phi(|y-k|) \le \Phi\left(\frac{|x-y|}{2}\right). \tag{2.5}$$

On the other hand, if $|y-k| < \frac{1}{2}|x-y|$, then

$$|x-k| \ge |x-y| - |y-k| > \frac{1}{2}|x-y|,$$

so that we have

$$\Phi(|x-k|) \le \Phi\left(\frac{|x-y|}{2}\right). \tag{2.6}$$

The estimations (2.5) and (2.6) then imply that

$$\begin{aligned}\sum_{k\in\mathbf{Z}} \Phi(|x-k|)\Phi(|y-k|) &= \left(\sum_{|y-k|\ge\frac{1}{2}|x-y|} + \sum_{|y-k|<\frac{1}{2}|x-y|}\right)\Phi(|x-k|)\Phi(|y-k|)\\ &\le \Phi\left(\frac{|x-y|}{2}\right)\left(\sum_{k\in\mathbf{Z}}\Phi(|x-k|) + \sum_{k\in\mathbf{Z}}\Phi(|y-k|)\right)\\ &\le 4\left(\Phi(0) + \int_0^\infty \Phi(t)dt\right)\Phi\left(\frac{|x-y|}{2}\right).\end{aligned}$$

Hence, by (2.3), we see that (2.4) holds with

$$C = 4\left(\Phi(0) + \int_0^\infty \Phi(t)dt\right). \quad \blacksquare$$

1. Proof of (a). For fixed $x \neq y$, we can find $j_0 \in \mathbb{Z}$ such that

$$2^{j_0} \le \frac{1}{|x-y|} < 2^{j_0+1}. \tag{2.7}$$

Setting

$$\Phi(x) = \frac{1}{(1+x)^{1+\epsilon}},$$

and applying the first inequality in (1.5), we see that

$$\begin{aligned} |K(x,y)| &\le C \sum_{j\le j_0} 2^j \sum_{k\in\mathbb{Z}} \Phi(|2^j x - k|)\Phi(|2^j y - k|) \\ &\quad + C \sum_{j>j_0} 2^j \sum_{k\in\mathbb{Z}} \Phi(|2^j x - k|)\Phi(|2^j y - k|) \\ &:= I_1 + I_2, \quad \text{say.} \end{aligned} \tag{2.8}$$

Now, since

$$\sum_k \Phi(|2^j x - k|)\Phi(|2^j y - k|) \le \Phi(0) + 2\int_0^\infty \Phi(t)dt,$$

we obtain

$$I_1 \le C \sum_{j \le j_0} 2^j \le C 2^{j_0}.$$

So, it follows from (2.7) that

$$I_1 \le C|x-y|^{-1}. \tag{2.9}$$

Also, if $j > j_0$, then (2.7) implies that

$$|2^j x - 2^j y| \ge 2^{j_0}|x-y| \ge 1/2.$$

Hence, by Lemma 2, we obtain

$$\begin{aligned} I_2 &\le C \sum_{j>j_0} 2^j (1 + |2^j x - 2^j y|)^{-1-\epsilon} \\ &\le C \sum_{j>j_0} 2^j \cdot 2^{-j(1+\epsilon)} |x-y|^{-1-\epsilon} \\ &\le C|x-y|^{-1}. \end{aligned} \tag{2.10}$$

By combining (2.8), (2.9) and (2.10), we obtain (a).

2. Proof of (b) and (c). To verify (b), we set

$$\eta = \frac{\varepsilon}{2(1+\alpha+\varepsilon)}$$

and observe that

$$\begin{aligned}
|K(x,y) - K(x',y)| &\le \sum_{j,k\in\mathbf{Z}} |\psi_{j,k}(x') - \psi_{j,k}(x)|^{\eta}|\psi_{j,k}(x') - \psi_{j,k}(x)|^{1-\eta}|\psi_{j,k}(y)| \\
&\le C\sum_{j,k\in\mathbf{Z}} |\psi_{j,k}(x') - \psi_{j,k}(x)|^{\eta}|\psi_{j,k}(x')|^{1-\eta}|\psi_{j,k}(y)| \\
&\quad + C\sum_{j,k\in\mathbf{Z}} |\psi_{j,k}(x') - \psi_{j,k}(x)|^{\eta}|\psi_{j,k}(x)|^{1-\eta}|\psi_{j,k}(y)| \\
&:= J_1 + J_2, \quad \text{say.}
\end{aligned} \tag{2.11}$$

The assumption (1.5) then leads to

$$\begin{aligned}
J_1 &\le C\sum_{j\in\mathbf{Z}}\{2^j|2^jx - 2^jx'|^{\alpha\eta}\sum_{k\in\mathbf{Z}}[\Phi(|2^jx'-k|)]^{1-\eta}\Phi(|2^jy-k|)\} \\
&= \left(C\sum_{j\le j_0} + C\sum_{j>j_0}\right)\{\dots\} \\
&:= J_{11} + J_{12}, \quad \text{say.}
\end{aligned} \tag{2.12}$$

Here, since

$$\sum_k[\Phi(|2^jx'-k|)]^{1-\eta}\Phi(|2^jy-k|) \le \Phi(0) + 2\int_0^\infty \Phi(t)dt,$$

we have

$$J_{11} \le C|x-x'|^{\alpha\eta}\sum_{j\le j_0} 2^{j(1+\alpha\eta)} \le \frac{C|x-x'|^{\alpha\eta}}{|x-y|^{1+\alpha\eta}}. \tag{2.13}$$

Hence, by Lemma 2, we see that, for $j > j_0$,

$$\sum_k[\Phi(|2^jx'-k|)]^{1-\eta}\Phi(|2^jy-k|) \le C\left[\Phi\left(\frac{1}{2}|2^jx' - 2^jy|\right)\right]^{1-\eta} \le \frac{C}{2^{j(1+\epsilon)(1-\eta)}|x-y|^{(1+\epsilon)(1-\eta)}}.$$

Thus, we arrive at

$$J_{12} \le C2^{j_0((1+\alpha\eta)-(1+\epsilon)(1-\eta))}\frac{|x-x'|^{\alpha\eta}}{|x'-y|^{(1+\epsilon)(1-\eta)}} \le C\frac{|x-x'|^{\alpha\eta}}{|x-y|^{1+\alpha\eta}}. \tag{2.14}$$

By combining (2.12)-(2.14), we obtain

$$J_1 \leq C\frac{|x - x'|^\delta}{|x - y|^{1+\delta}},$$

with $\delta = \alpha\eta$. Similarly, we also have

$$J_2 \leq C\frac{|x - x'|^\delta}{|x - y|^{1+\delta}}.$$

Hence, by applying (2.11), we establish the inequality (b). By symmetry, it is clear that (c) also holds.

3. Proof of (d). Recall from Lemma 1 that the linear operator T_I is bounded in L^2. It therefore follows that some $B > 0$ exists such that

$$\left\| \sum_{j,k\in\mathbf{Z}} a_{j,k}\psi_{j,k} \right\|_2^2 \leq B \sum_{j,k\in\mathbf{Z}} |a_{j,k}|^2$$

for any $\{a_{j,k}\} \in \ell^2$. By setting $a_{j,k} = \theta_{j,k}\langle f, \psi_{j,k}\rangle$, we obtain

$$\|T_\Theta f\|_2^2 \leq B \sum_{j,k\in\mathbf{Z}} |\theta_{j,k}|^2 |\langle f, \psi_{j,k}\rangle|^2 \leq B^2 \|\Theta\|_{\ell^\infty} \|f\|_2^2.$$

Hence, T_Θ is bounded in L^2. This completes the proof of Theorem 1. ■

In the proof of Theorem 1, we verify that under the assumptions (1.5) and (1.6), T_Θ is a Calderón-Zygmund operator. Hence, by a wellknown result on such operators (see Meyer [5], p. 230), statement (i) holds. In addition, assumption (1.6) implies that

$$T_\Theta^*(1) = T_\Theta(1) = 0.$$

Hence, both (ii) and (iii) also hold (see Meyer [5], p. 237 and p. 239).

To prove Theorem 2, we consider the operator

$$(P_\Theta f)(x) := \sum_{j,k\in\mathbf{Z}} \theta_{j,k}\langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}(x).$$

A similar discussion leads to the conclusion that the linear operator P_Θ is bounded in L^p This fact, in turn, implies that for any $f \in L^p$ and $\Theta \in \ell^\infty$ the series

$$\sum_{j,k\in\mathbf{Z}} \theta_{j,k}\langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}(x)$$

is convergent in L^p. Hence, $\{\psi_{j,k}(x)\}$ is an unconditional basis of L^p. ■

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To Jaap

FRACTAL FUNCTIONS AND SCHAUDER BASES

Z. Ciesielski (•)

1. Introduction. In recent years more and more attention is paid in mathematical papers to *fractal functions* and to *fractal sets*. There are various definitions of those objects. W assume that a compact set $K \in R^{d+1}$ is fractal, by definition, if its *box (entropy) dimension* $\dim_b(K) \neq j$ for $j = 0, 1, \ldots, d+1$ and $0 < \dim_b(K) < d+1$. In the same time the function $f : I^d \to R^d$, $I = [0,1]$, is *fractal*, by definition, if its graph $\Gamma_f = \{(\underline{t}, f(\underline{t})) : \underline{t} \in I^d\}$ has box dimension satisfying the inequalities $d < \dim_b(\Gamma_f) < d+1$. For the definitions and properties of lower $\underline{\dim}_b(K)$ and upper $\overline{\dim}_b(K)$ box (-counting) dimension we refer to [F]. In case $\underline{\dim}_b(K) = \overline{\dim}_b(K)$ by definition $\dim_b(K)$ is the common value.

The relation between box dimension of the graph of a function satisfying *Hölder condition* is known for years. In particular, it is known that the Hölder condition with some α, $0 < \alpha \leq 1$, i.e.

$$|f(\underline{t}) - f(\underline{t}')| \leq C \cdot |\underline{t} - \underline{t}'|^\alpha \quad \text{for} \quad \underline{t}, \underline{t}' \in I^d; \tag{1.1}$$

implies that

$$\dim_b(\Gamma_f) \leq d + 1 - \alpha. \tag{1.2}$$

Our aim is to describe some subclasses of functions f satisfying (1.1) for which equality takes place in (1.2). The Hölder classes, as it was shown in [C1], can be characterized by means of the coefficients of the Schauder basis expansions, and it seems natural to apply this tool to solve our problem.

In Section 2 we describe the constructions of the Schauder and Haar bases over cubes and state the main results on characterization of Hölder classes by means of the coefficients of the Schauder and Haar expansions. Senction 3 contains the main results on Hölder ubclasses for which we have equality in (1.2).

2. Haar and Schauder bases. The orthogonal Haar functions over I, normalized in the maximum norm, can be defined by means of the function sign(t). Define

$$h_0(t) = \frac{\text{sign}(t + \frac{1}{2}) - \text{sign}(t - \frac{1}{2})}{2},$$

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$$h_1(t) = \frac{\text{sign}(t+\frac{1}{2}) + \text{sign}(t-\frac{1}{2})}{2} - \text{sign}(t) \quad \text{for} \quad t \in R$$

and

$$h_{j,k}(t) = h_1\big(2^k(t - \frac{2j-1}{2^{k+1}})\big) \quad \text{where} \quad j = 1, \ldots, 2^k; \quad k = 0, 1, \ldots .$$

The Haar orthogonal system on I with respect to the Lebesgue measure is simply

$$\{1, h_{j,k},\ j = 1, \ldots, 2^k;\ k = 0, 1, \ldots\}.$$

We note also that

$$\text{supp}\ h_{j,k} = \big[\frac{(j-1)}{2^k}, \frac{j}{2^k}\big].$$

Often it is more convenient to index the Haar system as follows: $h_1 = 1$ and $h_n = h_{j,k}$ whenever $n = 2^k + j$ with some $j = 1, \ldots, 2^k; k = 0, 1, \ldots$

To define the *d-dimensional* orthogonal Haar functions over I^d properly we decompose at first the set of multi-indexes N^d, where $N = \{0, 1, \ldots\}$. Using the norm $|\underline{l}|_\infty = \max(l_1, \ldots, l_d)$ we introduce the decompositions

$$N^d = N_0 \cup \bigcup_{k \geq 0} N_k \quad \text{where} \quad N_k = \{\underline{l} : 2^k < |\underline{l}|_\infty \leq 2^{k+1}\},$$

N_0 contains $\underline{1} = (1, \ldots, 1)$ only and

$$N_k = \bigcup_{\emptyset \neq e \subset \mathcal{D}} N_{e,k} \quad \text{with} \quad \mathcal{D} = \{1, \ldots, d\},$$

where $N_{e,k} = \{\underline{l} \in N_k : 2^k < l_i \leq 2^{k+1} \text{ for } i \in e\}$. Now, the Haar orthogonal functions over I^d are defined as follows: $h_0(\underline{t}) = 1$ and for $\underline{l} \in N_{e,k}$

$$h_{\underline{l}}(\underline{t}) = \prod_{i \in e} h_{l_i - 2^k, k}(t_i) \prod_{i \in \mathcal{D}\setminus e} |h_{l_i, k}(t_i)|.$$

Thus, each $h_{\underline{l}}$, for $\underline{l} \in N_{e,k}$, has support which is a dyadic cube. Actually, over I^d we are given $2^d - 1$ functions orthogonal to 1 i.e. for each e, $\emptyset \neq e \subset \mathcal{D}$,

$$h_e(\underline{t}) = \prod_{i \in e} h_1(t_i) \prod_{i \in \mathcal{D}\setminus e} h_0(t_i)$$

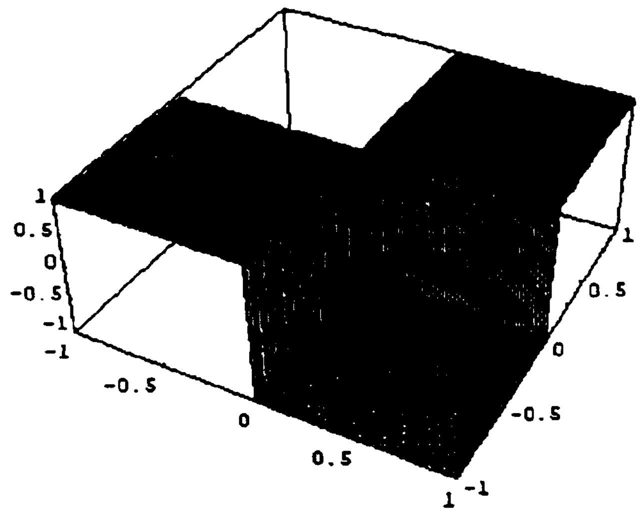
and for $\underline{l} \in N_{e,k}$

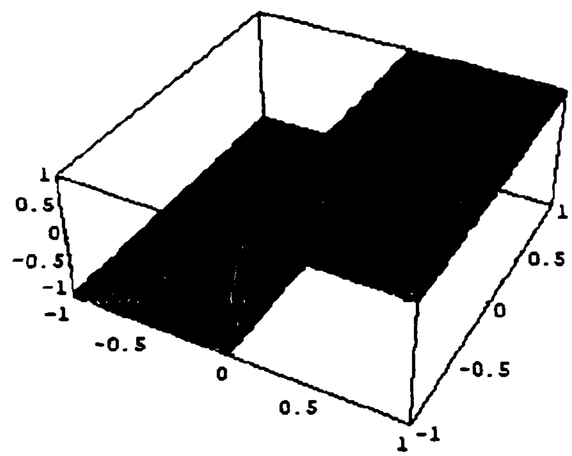
$$h_{\underline{l}}(\underline{t}) = h_e\big(2^k(\underline{t} - \frac{2\underline{j}-1}{2^{k+1}})\big),$$

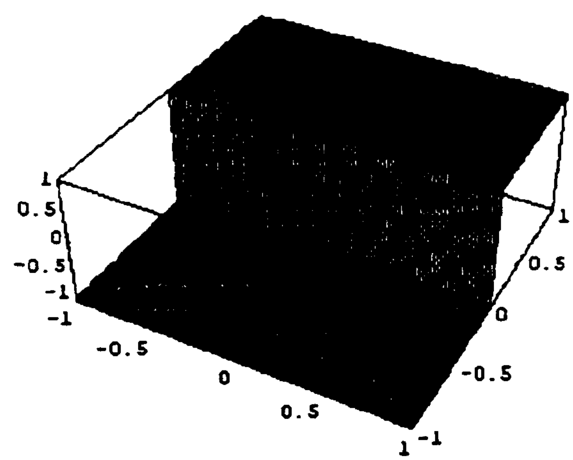
where $j_i = l_i - 2^k$ for $i \in e$ and $j_i = l_i$ for $i \in \mathcal{D}\setminus e$. Consequently, the support of $h_{\underline{l}}$ is the dyadic cube with center at $\frac{2\underline{j}-1}{2^{k+1}}$ and with edges of length $\frac{1}{2^k}$.

On the following page we present the graphs of the functions h_e in case $d = 2$.

Two dimensional Haar functions







The modulus of continuity of $f \in L^p(I^d)$ in the L^p space is defined by formula

$$\omega_p(f;\delta) = \sup_{0<|\underline{h}|<\delta} \left(\int_{I^d(\underline{h})} |f(\underline{t}+\underline{h}) - f(\underline{t})|^p d\underline{t} \right)^{\frac{1}{p}},$$

where $|h|$ is the eucleadian norm of $\underline{h}$ and $I^d(\underline{h}) = \{\underline{t} \in I^d : \underline{t} + \underline{h} \in I^d\}$. For later use we introduce the orthogonal projections

$$Q_0 f = (f, h_{\underline{0}})h_{\underline{0}}, \quad Q_k f = \sum_{\underline{j} \in N_k} \frac{(f, h_{\underline{j}})h_{\underline{j}}}{\|h_{\underline{j}}\|_2^2},$$

and

$$P_k f = Q_0 f + \cdots + Q_k f,$$

where

$$(f,g) = \int_{I^d} f(\underline{t})g(\underline{t})d\underline{t} \quad \text{and} \quad \|f\|_p = \left(\int_{I^d} |f(\underline{t})|^p d\underline{t} \right)^{\frac{1}{p}}.$$

It should be clear that over each dyadic cube of the $k-th$ generation in I^d the the function $Q_k f$ is constant and it is equal to the mean value of f over that particular dyadic cube. Thus, the Haar orthogonal system $\{h_{\underline{j}}\}$ has the norms $\|Q_k\|_p$, $1 \leq p \leq \infty$, bounded by 1. Consequently, the Haar system is a basis in the space L^p , $1 \leq p < \infty$. Moreover, we have

$$(2.1) \qquad (2^d - 1)^{-\frac{1}{p}} A_{k,p} \leq \| \sum_{\underline{n} \in N_k} a_{\underline{n}} \cdot h_{\underline{n}} \|_p \leq (2^d - 1)^{\frac{1}{p'}} A_{k,p},$$

where $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $a_{\underline{n}} \in R$, and

$$(2.2) \qquad A_{k,p} = \left(\frac{1}{2^{dk}} \sum_{\underline{n} \in N_k} |a_{\underline{n}}|^p \right)^{\frac{1}{p}}.$$

Moreover, we know from [C2]

Proposition 2.3. *Let $0 < \alpha < \frac{1}{p} \leq 1$ and let*

$$f \sim \sum_{\underline{n} \in N^d} a_{\underline{n}} \cdot h_{\underline{n}}.$$

Then

$$(2.4) \qquad \omega_p(f;\delta) = O(\delta^\alpha) \quad as \quad \delta \to 0_+$$

is equivalent to

$$(2.5) \qquad A_{k,p} = O(2^{-\alpha k}) \quad as \quad k \to \infty.$$

Moreover, for $f \in C(I^d)$, $0 < \alpha < 1$, and $p = \infty$, conditions (2.4) and (2.5) are equivalent.

To define the Schauder basis over I^d we start with the function $\psi(t) = \max[0, 1 - |t|]$ and with the set of D of all dyadic points in I. Define $D_0 = \{0, 1\}$, $D_k = \{\frac{2j-1}{2^k} : j = 1, \ldots, 2^{k-1}\}$ and $k = 1, 2, \ldots$. Thus

$$D = \bigcup_{k \geq 0} D_k,$$

and the Schauder functions over I are defined as follows

$$\phi_\tau(t) = \psi(2^k(t - \tau)) \quad \text{for} \quad \tau \in D_k, \ k = 0, 1, \ldots$$

For the Schauder functions over I^d it is convenient to introduce $C_0 = D_0$, $C_k = C_{k-1} \cup D_k$. Then

$$C_k^d = C_{k-1}^d \cup D_{k,d},$$

where

$$D_{k,d} = \{\underline{\tau} = (\tau_1, \ldots, \tau_d) \in C_k^d : \exists_i \ \tau_i \in D_k\} \text{ and } D_{0,d} = D_0^d.$$

Now,define

$$\phi_{\underline{\tau}}(\underline{t}) = \prod_{i \in \mathcal{D}} \psi(2^k(t_i - \tau_i)) \ \text{ for } \underline{\tau} \in D_{k,d}, \ k = 0, 1, \ldots \tag{2.6}$$

In the two dimensional case all the basic Schauder functions are obtainable, by suitable tanslations and rescaling, from functions presented by the pictures on the next page.

The system is called *the diamont* or *multi-affine* (cf. [R],[Se],[Sh]) basis in the Banach space $C(I^d)$. Some of its properties we mention here. Like in the Haar case we have with some constant C depending on the dimension only, for $1 \leq p \leq \infty$, the inequalities

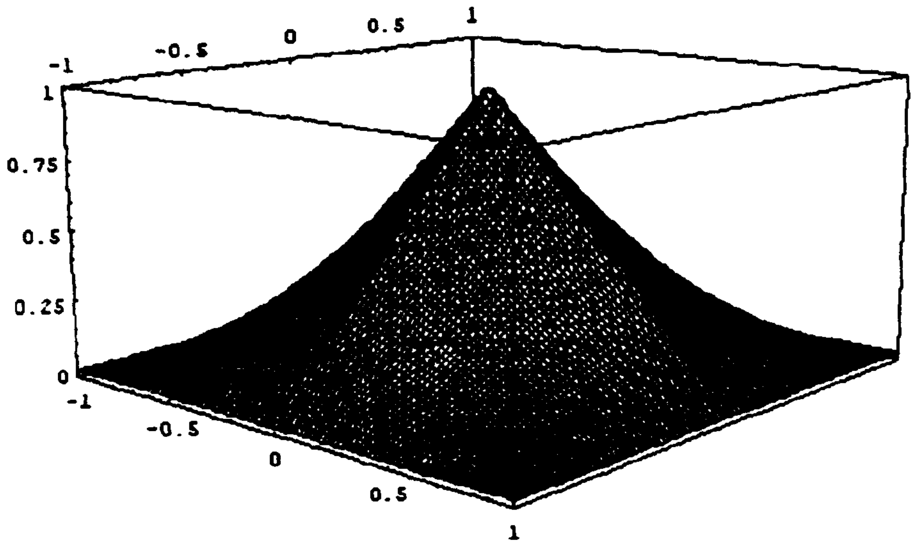
$$p\frac{1}{C} \cdot B_{k,p} \leq \| \sum_{\underline{\tau} \in D_{k,d}} b_{\underline{\tau}} \cdot \phi_{\underline{\tau}} \|_p \leq C \cdot B_{k,p}, \tag{2.7}$$

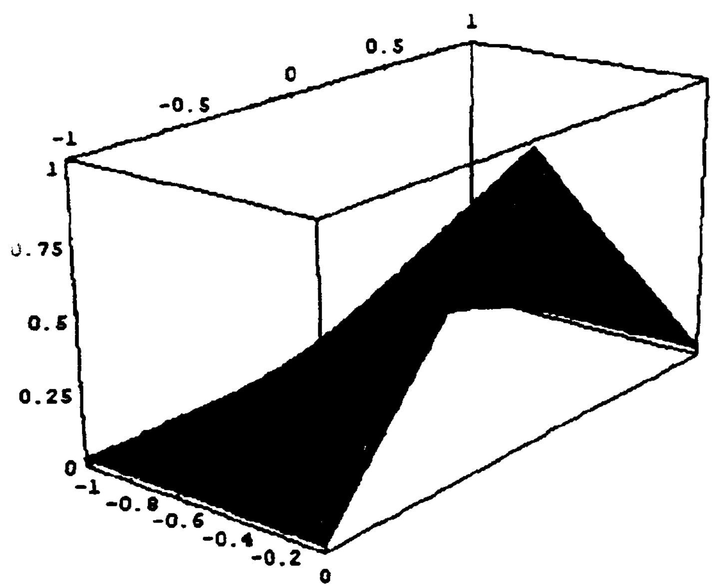
with

$$B_{k,p} = \left(\frac{1}{|D_{k,d}|} \sum_{\underline{\tau} \in D_{k,d}} |b_{\underline{\tau}}|^p \right)^{\frac{1}{p}}, \tag{2.8}$$

where $|D_{k,d}|$ is the cardinality of $D_{k,d}$.

Two dimensional diamont functions





The biorthogonal to $(\phi_{\underline{\tau}}(\underline{t}),\ \underline{\tau} \in D^d)$ system of linear functionals over $C(I^d)$ is known (see e.g. [R]) and for given $f \in C(I^d)$ and $\underline{\tau} \in D^d$ the corresponding functionals are defined as follows:

$$b_{\underline{\tau}}(f) = f(\underline{\tau}) \quad \text{for} \quad \underline{\tau} \in D_{0,d},$$

and for $k \geq 1$

$$b_{\underline{\tau}}(f) = \frac{1}{2^d} \sum_{\epsilon \in \{-1,1\}^d} (f(\underline{\tau}) - f(\underline{\tau}^\epsilon)) \quad \text{for} \quad \tau \in D_{k,d},$$

where $\underline{\tau}^\epsilon = (\underline{\tau}_1^\epsilon, \ldots, \underline{\tau}_d^\epsilon)$ with

$$\tau_i{}^\epsilon = \begin{cases} \tau_i + \epsilon_i \cdot 2^{-k} & \text{if } \tau_i \in D_k; \\ \tau_i & \text{if } \tau_i \in C_{k-1}. \end{cases}$$

It is convenient to introduce the finite dimensional projections in the space $C(I^d)$

$$R_k(f) = \sum_{\underline{\tau} \in D_{k,d}} b_{\underline{\tau}}(f) \cdot \phi_{\underline{\tau}}.$$

The fact that $(\phi_{\underline{\tau}}(\underline{t}),\ \underline{\tau} \in D^d)$ is a Schauder basis in $C(I^d)$ can now be stated as follows: for each $f \in C(I^d)$ the series

$$\sum_{k=0}^{\infty} R_k(f)$$

converges to f in the maximum norm. Finally we state the main property (c.f. [C1], [R],[Sh])

Proposition 2.9. *Let $0 < \alpha < 1$, $f \in C(I^d)$, and let*

$$f = \sum_{\underline{\tau}} b_{\underline{\tau}} \phi_{\underline{\tau}}.$$

Then , the following conditions are equivalent :

(i)
$$\omega_\infty(f;\delta) = O(\delta^\alpha),$$

(ii)
$$\max_{\underline{\tau} \in D_{k,d}} |b_{\underline{\tau}}| = O(2^{-\alpha k}),$$

(iii)
$$\|f - \sum_{i \leq k} R_i(f)\|_\infty = O(2^{-\alpha k}).$$

3. Box dimension of graphs. In this section we are going to apply the Haar and Schauder bases to compute the box dimension $\dim_b(\Gamma_f)$ for some reasonable subclasses of the Hölder classes on cubes.

Theorem 3.1. *Let $0 < \alpha \leq \beta \leq 1$ and let the function f be given on I^d by the Haar series*

$$f = \sum_k \sum_{\underline{n} \in N_k} a_{\underline{n}} \cdot h_{\underline{n}}.$$

If

$$A_{k,\infty} = \max_{N_k} |a_{\underline{n}}| = O(\frac{1}{2^{\alpha k}}),$$

then

$$\overline{\dim}_b(\Gamma_f) \leq d + 1 - \alpha.$$

Moreover , if for some $C > 0$,

$$A_{k,1} = \frac{1}{2^{kd}} \sum_{N_k} |a_{\underline{n}}| \geq C \cdot \frac{1}{2^{\beta k}},$$

then

$$\underline{\dim}_b(\Gamma_f) \geq d + 1 - \beta.$$

Corollary 3.2. *If there is a positive finete constant C such that*

$$\frac{1}{C \cdot 2^{\beta k}} \leq \frac{1}{2^{kd}} \sum_{N_k} |a_{\underline{n}}| \leq \max_{N_k} |a_{\underline{n}}| \leq C \cdot \frac{1}{2^{\alpha k}},$$

then

$$d + 1 - \beta \leq \dim_b(\Gamma_f) \leq d + 1 - \alpha.$$

Note, no continuity of f is assumed in this statement.

Theorem 3.3. *Let $0 < \alpha \leq \beta \leq 1$ and let the function f be given on I^d by the Schauder series*

$$f = \sum_k \sum_{\underline{r} \in D_{k,d}} b_{\underline{r}} \cdot \phi_{\underline{r}}.$$

If

$$B_{k,\infty} = \max_{D_{k,d}} |b_{\underline{r}}| = O(\frac{1}{2^{\alpha k}}),$$

then

$$\overline{\dim}_b(\Gamma_f) \leq d + 1 - \alpha.$$

Moreover, if for some $C > 0$,

$$B_{k,1} = \frac{1}{|D_{k,d}|} \sum_{D_{k,d}} |b_{\underline{r}}| \geq C \cdot \frac{1}{2^{\beta k}},$$

then

$$\underline{\dim}_b(\Gamma_f) \geq d + 1 - \beta.$$

Corollary 3.4. *If there is a positive finete constant C such that*

$$\frac{1}{C \cdot 2^{\beta k}} \leq \frac{1}{|D_{k,d}|} \sum_{D_{k,d}} |b_{\underline{\tau}}| \leq \max_{D_{k,d}} |b_{\underline{\tau}}| \leq C \cdot \frac{1}{2^{\alpha k}},$$

then

$$d + 1 - \beta \leq \dim_b(\Gamma_f) \leq d + 1 - \alpha.$$

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Integrals of subharmonic functions along two curves

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Dedicated to Jaap Korevaar on his 70th birthday

1 Introduction

Suppose that C_0 is the unit circle $|z| = 1$ and that C is another rectifiable curve, whose interior D contains $|z| < 1$. We also suppose that C is not identical with C_0. The length L of C and the area A of D are related by the isoperimetric inequality

$$L^2 \geq 4\pi A > 4\pi^2, \qquad \text{i.e. } L > 2\pi.$$

Thus

$$\int_C |dz| > \int_{C_0} |dz|.$$

This makes it reasonable to ask the following question. Suppose that $\mathcal{F}$ is a class of functions $u(z)$ defined in $\bar{D}$. Does there exist a constant $K = K(\mathcal{F}, C)$, such that, when $u \in \mathcal{F}$,

$$\int_{C_0} u(z)|dz| \leq K \int_C u(z)|dz|. \tag{1}$$

We denote by $K(\mathcal{F}, C)$ the smallest such constant K. If K does not exist we write $K(\mathcal{F}, C) = \infty$.

The above problem was raised by Harold Shapiro [9] in a lecture in York about 1982 for the class $\mathcal{P} = \{u \mid u(z) = |P(z)|\}$, where P is a polynomial. We prove Shapiro's conjecture in a companion paper [5] with $K = e(4/\pi + 3) < 12$. The first proof with an absolute but unspecified constant was given by Garnett, Gehring and Jones [4]. More general results were obtained by Fernandez and Hamilton [3] and Bishop and Jones [1]. In the present paper we are concerned with the wider class

$$\begin{aligned} \mathcal{S} \;=\; & \{u \mid u(z) \text{ is subharmonic (s.h.) in } D \text{ and} \\ & \text{upper semicontinuous (u.s.c.) in } \bar{D}\} \end{aligned}$$

for which the analogue of Shapiro's conjecture does not hold in general. We also define

$$\begin{aligned}
\mathcal{S}^+ &= \{u \mid u \in \mathcal{S} \text{ and } u \geq 0 \text{ in } D\}, \\
\mathcal{H} &= \{u \mid u(z) \text{ is harmonic in } D \text{ and continuous in } \bar{D}\}, \\
\mathcal{H}^+ &= \{u \mid u \in \mathcal{H} \text{ and } u \geq 0 \text{ in } \bar{D}\}, \\
\mathcal{A} &= \{v \mid v = e^u, \text{ where } u \in \mathcal{S}\}.
\end{aligned}$$

It will turn out that we can replace $\mathcal{S}$ by $\mathcal{S}^+$, $\mathcal{H}$ or $\mathcal{H}^+$ in the definition of $\mathcal{A}$ without altering the constant $K(\mathcal{A}, C)$ and also that $K(\mathcal{A}, C) = K(\mathcal{P}, C)$. On the other hand for the classes $\mathcal{F} = \mathcal{S}^+$ or $\mathcal{H}^+$ the constant $K(\mathcal{F}, C)$ may be infinite.

The following result was essentially discovered by T. Sheil-Small soon after Shapiro's lecture but never published by him

Theorem 1 *If C is convex $K(\mathcal{S}^+, C) < 2$. Given $\epsilon > 0$, there exists a circle C, such that $K(\mathcal{S}^+, C) > 2 - \epsilon$.*

The example of circles shows that the situation is indeed different for $\mathcal{S}^+$ and $\mathcal{A}$ or $\mathcal{P}$.

Theorem 2 *If C is a circle, then $K(\mathcal{A}, C) \leq 1$, with equality only when C touches C_0.*

If C does not touch C_0 and the domain bounded by C, C_0 is conformally equivalent to an annulus $1 < |w| < R$, we have $K(\mathcal{A}, C) = 1/R$.

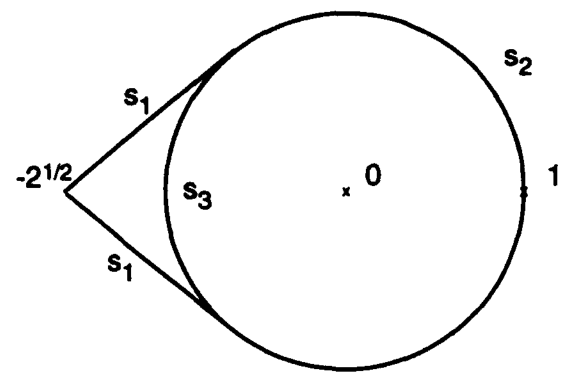
Theorem 2 raises the question of whether (1) holds with $K \leq 1$ for a wider class of functions and curves. This conclusion is false in general even for a linear polynomial and a convex curve.

Example 1.

Suppose that C is the convex hull of the unit circle together with the point $z = -\sqrt{2}$. Thus C consists of the straight line segments s_1: from $-\sqrt{2}$ to $e^{\pm 3\pi i/4}$ together with the arc s_2: $z = e^{i\theta}$, $|\theta| \leq 3\pi/4$ of C_0. Let s_3 be the arc s_3: $z = e^{i\theta}$, $3\pi/4 \leq \theta \leq 5\pi/4$ of C_0. Then the image of s_1 by $w = 1 + z\sqrt{2} + z^2/2 = (z + \sqrt{2})^2/2 = P(z)$ say is the line segment $[-i/2, i/2]$, while the image of s_3 is an arc joining $\pm i/2$, which is not a line segment. Thus

$$1 = \int_{s_1} |z + \sqrt{2}||dz| = \int_{s_1} |dw| < \int_{s_3} |dw| = \int_{s_3} |z + 2\sqrt{2}||dz|,$$

and $K(\mathcal{A}, C) > 1$.



The question of the exact value of $K(\mathcal{A}, C)$ seems to be quite complicated if C is not a circle. However the following result settles the value of $K(\mathcal{S}^+, C)$ in terms of a conformal mapping.

Theorem 3 *Suppose that $w = \phi(z)$ maps $|w| < 1$ (1,1) conformally onto D, so that $\phi(0) = 0$. Since C is rectifiable $\phi(z)$ has an absolutely continuous extension $\phi(e^{i\theta})$ to $|z| = 1$ and*

$$K(\mathcal{S}^+, C)^{-1} = \inf_{0\le\theta\le 2\pi} \left|\frac{\partial}{\partial\theta}\phi(e^{i\theta})\right|, \tag{2}$$

where inf *denotes the essential infimum, i.e. outside sets of measure zero.*

Conditions for $|\phi'(z)|$ to be bounded below are given for instance in Pommerenke's book [7, Corollary 10.2. p.308]. It is sufficient that there exists a fixed Dini-smooth curve Γ with a point P_0 on Γ such that, for every P on C, there is a rotation and translation Γ' of Γ which lies outside D, but such that the image P_0' of P_0 coincides with P. On the other hand this condition is "almost necessary". In particular if, at some point P, C has an interior angle of opening greater than π, we have $K(\mathcal{S}^+, C) = \infty$.

Since $K(\mathcal{A}, C) < \infty$, but $K(\mathcal{S}^+, C)$ may be infinite we may ask the corresponding question for the classes

$$\mathcal{S}^{+p} = \{v \mid v = u^p, \text{ where } u \in \mathcal{S}^+\}, \text{ and } 1 \le p < \infty.$$

Example 2. If C is the circle $|w+2| = 4$, together with the segment s: $[-6, -2]$, then $K(\mathcal{S}^{+p}, C) = \infty$ for $p \le 2$.

We note that if $p < q$, then $\mathcal{S}^{+q} \subset \mathcal{S}^{+p}$. For if $v = u^q$, where u is s.h. in D, then $v = (u^{q/p})^p$. Also $u^{q/p}$ is s.h. in D [6, Corollary 1, p.46]. Thus $v \in \mathcal{S}^{+p}$, so that $K(\mathcal{S}^{+p}, C) \ge K(\mathcal{S}^{+q}, C)$. Hence it is enough to consider the case $p = 2$.

We now define

$$\begin{aligned} u_\epsilon(x) &= |2+x|^{-1/2}|\log|2+x||^{-3/4}, \qquad -5/2 \le x \le -2-\epsilon \\ u_\epsilon(x) &= 0, \qquad \text{elsewhere on } C, \end{aligned}$$

where $0 < \epsilon < 1/4$. We extend u_ϵ as a bounded harmonic function to the disk $|w+2| < 4$ cut alonq s. Then $u_\epsilon \in \mathcal{S}^+$, so that $v_\epsilon = u_\epsilon^2 \in \mathcal{S}^{+2}$. Also

$$\begin{aligned} \int_C v_\epsilon(z)|dz| &= \int_\epsilon^{1/2} t^{-1}(\log 1/t)^{-3/2}dt \\ &< \int_0^{1/2} t^{-1}(\log 1/t)^{-3/2}dt = 2(\log 2)^{-1/2}. \end{aligned}$$

But if $\omega(r)$ is the harmonic measure at the origin of the segment $[-r,-2]$, then[1] $\omega(r) = \frac{2}{\pi}\left\{(\tan^{-1}\sqrt{r-2} - \tan^{-1}\frac{1}{4}\sqrt{r-2}\right\} \approx \frac{3}{2\pi}\sqrt{r-2}$ as $r \to 2$. thus we have as $\epsilon \to 0$

$$u_\epsilon(0) = \int_2^{5/2} u_\epsilon(-r)d\omega(r) \to \int_2^{5/2} \frac{1}{(r-2)^{1/2}|\log(r-2)|^{3/4}} \frac{3dr}{4\pi(r-2)^{1/2}} = \infty.$$

Thus by Schwarz's inequality

$$\begin{aligned} \int_0^{2\pi} v_\epsilon(e^{i\theta})d\theta &= \int_0^{2\pi} u_\epsilon(e^{i\theta})^2 d\theta \ge \frac{1}{2\pi}\left\{\int_0^{2\pi} u_\epsilon(e^{i\theta})d\theta\right\}^2 \\ &= 2\pi u_\epsilon(0)^2 \to \infty \qquad \text{as } \epsilon \to 0. \end{aligned}$$

Thus (1) cannot hold for any K. We note that C is not a Jordan curve, but by approximating C from inside the cut disk we can obtain a Jordan curve C' with the corresponding property.

Finally we state the following result which was mentioned earlier

Theorem 4 *We have for every rectifiable curve C*

$$K(\mathcal{S}^+, C) = K(\mathcal{H}^+, C) \tag{3}$$

and

$$K(\mathcal{A}, C) = K(\mathcal{P}, C). \tag{4}$$

[1] cf. Lemma 1 below.

2 Proof of Theorem 4

We suppose that (1) holds for all u in $\mathcal{H}^+(C)$ with a certain constant K and deduce that (1) also holds in the larger class $\mathcal{S}^+(C)$ with the same K. Suppose then that $u \in \mathcal{S}^+(C)$. Then $\psi(\zeta) = u(\zeta)$ is u.s.c. and $\psi(\zeta) \geq 0$ on C. Thus [6, p.5] there exist on C functions $\psi_n(\zeta)$ continuous w.r.t. ζ and strictly decreasing with n such that

$$\psi_n(\zeta) \to \psi(\zeta) \qquad \text{as } n \to \infty.$$

We denote by $\mathcal{F}(C, \psi)$ the class of functions u in $\mathcal{S}^+(C)$ such that

$$u(\zeta) \leq \psi(\zeta) \qquad \text{on } C.$$

Let $U_n(\zeta)$ be the harmonic extension of $\psi_n(\zeta)$ into D [6, p.70]. Thus $U_n(z) \in \mathcal{H}^+(C)$. By the maximum principle $U_n(z)$ decreases with n in D and, since $U_n(z) \geq 0$, $U_n(z) \to U(z)$, where $U(z)$ is harmonic in D [6, p.37]. We have

$$u(\zeta) = \psi(\zeta) < \psi_n(\zeta) = U_n(\zeta) \text{ on } C.$$

Thus by the maximum principle

$$u(z) \leq U_n(z) \text{ in } D.$$

Letting n tend to ∞ we deduce that $u(z) \leq U(z)$ in D. Also $U \in \mathcal{F}(C, \psi)$. In fact $U(z)$ is harmonic in D and $U(z) \leq U_n(z)$ so that, if $\zeta \in C$,

$$\overline{\lim_{z\to\zeta}} U(z) \leq \overline{\lim_{z\to\zeta}} U_n(z) = \psi_n(\zeta).$$

This is true for every n so that

$$\overline{\lim_{z\to\zeta}} U(z) \leq \psi(\zeta).$$

If we define $U_0(z)$ in $\bar{D}$ by

$$U_0(z) = U(z) \text{ in } D, \qquad U_0(\zeta) = \psi(\zeta) \text{ on } C,$$

we see that $U_0(z)$ is u.s.c. in $\bar{D}$, harmonic in D and $U_0(\zeta) \leq \psi(\zeta)$ on C, so that $U_0(z) \in \mathcal{F}(C, \psi)$, while if $u \in \mathcal{F}(C, \psi)$ we have $u(z) \leq U_0(z)$ in $\bar{D}$.

Thus by hypothesis, and since $U_n(z) \in \mathcal{H}^+(C)$ we have

$$\int_{C_0} u(z)|dz| \leq \int_{C_0} U(z)|dz| \leq \int_{C_0} U_n(z)|dz| \leq K \int_C U_n(\zeta)|d\zeta| = K \int_C \psi_n(\zeta)|d\zeta|.$$

Letting n tend to ∞, we deduce that

$$\int_{C_0} u(z)|dz| \leq K \int_{C_0} \psi(\zeta)|d\zeta| = K \int_{C_0} u(\zeta)|d\zeta|.$$

Thus (1) holds for all functions in $\mathcal{S}^+(C)$ if it holds for all functions in $\mathcal{H}^+(C)$. The converse is obvious, since $\mathcal{H}^+(C) \subset \mathcal{S}^+(C)$. This proves (3).

We next prove (4). Suppose that (1) holds for all functions of the form

$$u(z) = e^{V(z)} \tag{5}$$

where $V(z) \in \mathcal{H}^+(C)$. By considering $V(z) - m$ instead of $V(z)$, where m is the lower bound of $V(z)$ in $\bar{D}$, we deduce that (1) also holds whenever $V(z) \in \mathcal{H}(C)$.

We proceed to show that the conclusion also holds whenever $V(z) \in \mathcal{S}(C)$. The argument is similar to that given for the proof of (3). Suppose in fact that $v(z) \in \mathcal{S}(C)$. Let $\psi_n(\zeta)$ be a sequence of continuous functions decreasing to $v(z)$ on C. Let $V_n(z)$ be the harmonic extensions of $\psi_n(\zeta)$ to D. Then by hypothesis (1) holds for $U_n(z) = e^{V_n(z)}$. By the maximum principle we have $v(z) \leq V_n(z)$ and so $u(z) = e^{v(z)} \leq U_n(z)$ in D. We deduce that

$$\int_{C_0} u(z)|dz| \leq \int_{C_0} U_n(z)|dz| \leq K \int_C e^{\psi_n(z)}|dz|.$$

Letting n tend to ∞ we obtain

$$\int_{C_0} u(z)|dz| \leq K \int_C e^{v(z)}|dz| = K \int_C u(z)|dz|.$$

Thus (1) holds whenever $\log u(z) \in \mathcal{S}(C)$ if it holds whenever $\log u \in \mathcal{H}^+(C)$. This shows that we obtain the same constant $K(\mathcal{A}, C)$ if we replace $\mathcal{S}$ by $\mathcal{S}^+$, $\mathcal{H}$ or $\mathcal{H}^+$ in the definition of $\mathcal{A}$. If $P(z)$ is a polynomial then $\log|P(z)|$ is s.h. in the plane and this yields $K(\mathcal{A}, C) \geq K(\mathcal{P}, C)$.

To complete the proof of (4) we now show that $K(\mathcal{A}, C) \leq K(\mathcal{P}, C)$. To do this suppose that (1) holds with some constant K whenever $u(z) = |P(z)|$ where P is a polynomial. We shall deduce that (1) still holds when

$$u(z) = e^{h(z)} \tag{6}$$

where $h(z) \in \mathcal{H}(C)$ and so, by what we proved above, whenever $u(z) \in \mathcal{A}$. Suppose then that $h(z) \in \mathcal{H}(C)$. Then, by a theorem of Walsh [11], given a positive ϵ, there exists $h_1(z)$ harmonic in a simply connected neighbourhood N of $\bar{D} = D \cup C$, such that

$$|h_1(z) - h(z)| < \epsilon \text{ on } \bar{D}.$$

The function $h_1(z)$ has a harmonic conjugate $h_2(z)$ in N, so that

$$f(z) = \exp(h_1 + ih_2) \tag{7}$$

is regular in N and in particular on $\bar{D}$. Thus by a further theorem of Walsh [10] we can, given $\eta > 0$, find a polynomial $P(z)$, such that

$$|f(z) - P(z)| < \eta \text{ on } \bar{D}.$$

Let m, M be the minimum and maximum respectively of $|f(z)| = \exp(h_1)$ on $\bar{D}$. We assume that $4\eta/m < \epsilon < 1$, so that $|P(z)| > m/2$ on $\bar{D}$. Also

$$||f(z)/P(z)| - 1| \leq |f(z)/P(z) - 1| < \eta/|P(z)| < 2\eta/m < \epsilon/2 < 1/2.$$

Thus on $\bar{D}$ we have

$$|\log|f(z)/P(z)|| < 2||f(z)/P(z)| - 1| < \epsilon,$$

i.e.

$$e^{-\epsilon}e^{h_1(z)} < |P(z)| < e^{\epsilon}e^{h_1(z)}.$$

On combining this with (6) and (7) we obtain

$$e^{-2\epsilon}u(z) < |P(z)| < e^{2\epsilon}u(z).$$

Since (1) holds for $P(z)$ we deduce that

$$\int_{C_0} u(z)|dz| \leq e^{2\epsilon}\int_{C_0}|P(z)||dz| \leq Ke^{2\epsilon}\int_C |P(z)||dz| \leq Ke^{4\epsilon}\int_C u(z)|dz|.$$

Since ϵ is an arbitrary positive number, we deduce that $u(z)$ satisfies (1) so that (4) holds and Theorem 4 is proved.

3 Proof of Theorem 3

Suppose that C is a rectifiable curve and that

$$w = \phi(z) = \sum_{n=1}^{\infty} a_n z^n$$

maps $|z| < 1$ (1,1) conformally onto D. Then by a Theorem of F. and M. Riesz ([8] see also [2, Theorem 3.3, p.50]) $\phi(z)$ has a continuous extension to $|z| \leq 1$ and $\phi(e^{i\theta})$ yields an absolutely continuous parametrisation of C. Let $z = \psi(w) = \phi^{-1}(w)$ be the inverse map, let Γ be an arc of C and let $\omega(w, \Gamma)$ be the harmonic measure of Γ. Thus ω is harmonic in D, bounded in $\bar{D}$ and continuous except at the two endpoints of Γ. Also $\omega = 1$ at interior points of Γ and $\omega = 0$ at interior points of $\mathbf{C} \setminus \Gamma$.

The harmonic measure of the corresponding arc

$$z = e^{i\theta}, \qquad \theta_1 \le \theta \le \theta_2$$

of C_0 is given by

$$= \frac{1}{\pi}\left\{\arg\left(\frac{z - e^{i\theta_2}}{z - e^{i\theta_1}}\right) + \frac{\theta_1 - \theta_2}{2}\right\},$$

since this function is clearly harmonic and bounded in $|z| < 1$ and has the right boundary values. Also since harmonic measure is conformally invariant it follows that

$$\omega(w, \Gamma) = \frac{1}{\pi}\mathrm{Im}\ \log\left\{\frac{\psi(w) - \psi(w_2)}{\psi(w) - \psi(w_1)}\left(\frac{\psi(w_1)}{\psi(w_2)}\right)^{1/2}\right\} \tag{8}$$

where w_1, w_2 are the endpoints of Γ, when Γ is described in the anti-clockwise sense.

We recall that if $u(w) \in \mathcal{H}^+(C)$ then, for $w \in D$, $u(w)$ has the representation [6, p.114]

$$u(w) = \int_C u(W) d\omega(w, W).$$

Here $\omega(w, W)$ is the harmonic measure of the arc (W_0, W) of C, where W_0 is a fixed point and W a variable point of C. Also, since $u(w)$ is harmonic in $|w| < 1$ and continuous in $|w| \le 1$ we have

$$\int_{C_0} u(z)|dz| = \int_0^{2\pi} u(e^{i\theta})d\theta = 2\pi u(0) = 2\pi \int_C u(W)d\omega(0, W).$$

We now deduce from (8), writing $w = 0$ and $w_2 = W$ that, almost everywhere on C

$$\begin{aligned} d\omega(0, W) &= \frac{1}{\pi}\mathrm{Im}\ d\log\left\{\psi(W)^{1/2}\right\} = \frac{1}{2\pi}\mathrm{Im}\ \frac{\psi'(W)}{\psi(W)}dW \\ &= \frac{1}{2\pi i}\frac{\psi'(W)}{\psi(W)}dW = \frac{1}{2\pi}|\psi'(W)||dW|, \end{aligned}$$

since $|\psi(W)|$ and so $\log|\psi(W)|$ is constant on C. Thus

$$2\pi \int_C u(W)d\omega(0, W) = \int_C u(W)|\psi'(W)||dW| \le K \int_C u(W)|dW|,$$

where

$$K = \sup_{w \in C} |\psi'(w)| = \sup_{z \in C_0} \frac{1}{|\phi'(z)|}. \tag{9}$$

Here sup denotes the essential supremum, i.e. ignoring sets of measure zero. This proves (1) with K given by (9), i.e. (2).

It remains to show that the inequality is sharp. We recall that $\phi(z)$ has a continuous extension to $|z| \leq 1$, that $\phi(e^{i\theta})$ is absolutely continuous and that the length of the arc $\{\phi(e^{i\theta_1}), \phi(e^{i\theta_2})\}$ is given by

$$l(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} |\phi'(e^{i\theta})| d\theta.$$

Let μ be the essential infimum of $|\phi'(z)|$ on $|z| = 1$. Then

$$|\phi'(e^{i\theta})| \geq \mu$$

almost everywhere but $|\phi'(e^{i\theta})| < \mu + \epsilon$ on a set of positive measure. Since $|\phi'(e^{i\theta})|$ is the derivative of its integral almost everywhere we can find θ_0 and a positive h such that $|\phi'(e^{i\theta_0})| < \mu + \epsilon$ and

$$l(\theta_0 - h, \theta_0 + h) = \int_{\theta-h}^{\theta+h} |\phi'(e^{i\theta})| d\theta < (\mu + \epsilon)2h. \tag{10}$$

We write $\theta_1 = \theta_0 - h$, $\theta_2 = \theta_0 + h$, $w_1 = \phi(e^{i\theta_1})$, $w_2 = \phi(e^{i\theta_2})$ and define the function u to have boundary values 1 on the arc w_1, w_2 of C and zero on the complementary arc of C. If u is the bounded harmonic extension of these values onto D, then u is harmonic in D, $0 \leq u \leq 1$ in $\bar{D}$ and u is u.s.c. in $\bar{D}$ if we define $u(w_1) = u(w_2) = 1$. Thus $u \in S^+(C)$. Also

$$\int_C u(w)|dw| = \int_{w_1}^{w_2} |dw| = l(\theta_1, \theta_2) < (\mu + \epsilon)2h$$

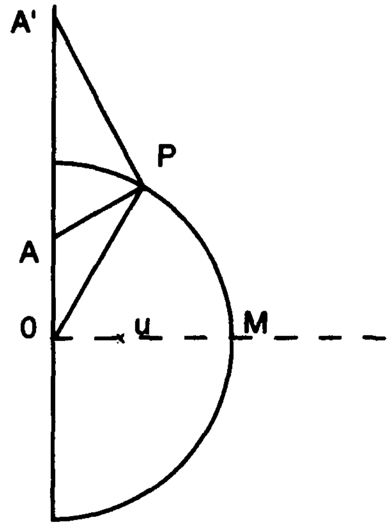
by (10), while

$$\int_{C_0} u(w)|dw| = 2\pi u(0) = 2h.$$

To see this we note that $\omega(w)$, given by (8) with $w_1 = e^{i\theta_1}$, $w_2 = e^{i\theta_2}$ has the correct boundary values and so coincides with $u(w)$. Thus

$$\int_C u(w)|dw| < (\mu + \epsilon) \int_{C_0} u(w)|dw|.$$

Here ϵ is an arbitrary positive number. Thus (1) cannot hold with $K < 1/\mu$ if $\mu > 0$ or with any positive constant if $\mu = 0$. This completes the proof of Theorem 3.



4 Proof of Theorem 1

Theorem 1 could be deduced from Theorem 3 but it is as easy to give a direct proof. We need a Lemma on harmonic measure in a semidisk.

Lemma 1 *Suppose that D is the semidisk*

$$|w| < M, u > 0, \text{ where } w = u + iv.$$

Then if E is a set on the imaginary axis of length l and $\omega = \omega(E, D)$ is the harmonic measure w.r.t. D of E at a point u, where $0 < u < M$, we have

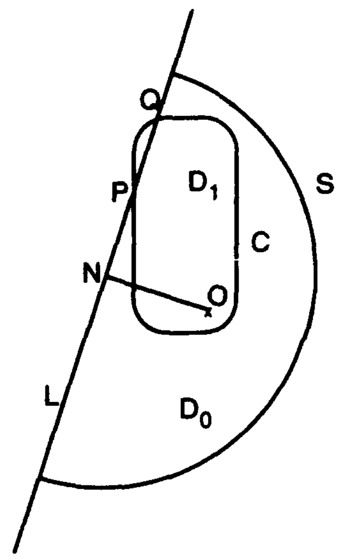
$$\omega \leq \frac{l}{\pi u}\left\{1 - \left(\frac{u}{M}\right)^2\right\}.$$

Suppose first that E is the segment OA of the imaginary axis, where O is the origin, A the point ia and $0 < a < M$, and let A' be the point iM^2/a. Then if P is any point of D, we have

$$\omega(P, E) = \frac{1}{\pi}\{\text{ angle } OPA - \text{ angle } OA'P\}.$$

In fact the right hand side is 1 on the segment OA and zero at other points of the segment $\{-iM, iM\}$. If P lies on $|w| = M$ the triangles OPA and $OA'P$ are similar. Thus $\omega(P, E)$ vanishes on the semicircle $|\omega| = M$, $u > 0$ also and so has the required boundary values. If P is the point u on the real axis we deduce that

$$\omega(P) = \frac{1}{\pi}\left\{\tan^{-1}\left(\frac{a}{u}\right) - \tan^{-1}\left(\frac{ua}{M^2}\right)\right\}.$$



Differentiating w.r.t. a, we see that a small segment of length δa at a has harmonic measure approximately

$$\delta\omega = \frac{\delta a}{\pi}\left\{\frac{u}{a^2+u^2} - \frac{M^2u}{a^2u^2+M^4}\right\} < \frac{\delta a}{\pi u}\left\{1-\left(\frac{u}{M}\right)^2\right\},$$

since

$$\frac{u^2}{a^2+u^2} - \frac{M^2u^2}{a^2u^2+M^4} - \frac{M^2-u^2}{M^2} = \frac{a^2(u^6-M^6)+a^4u^2(u^2-M^2)}{M^2(a^2+u^2)(a^2u^2+M^4)} < 0.$$

Integrating the characteristic function of E w.r.t. $d\omega$ and da we deduce Lemma 1.

To complete the proof of Theorem 1, we suppose that the closed curve C contains $|w| < 1$ and lies in $|w| \leq M/2$, so that $M \geq 2$. Let P, Q be two neighbouring points of C and let L be the line through PQ. Let N be the foot of the perpendicular from the origin O to L and let D_0 be the semidisk which contains the origin and is bounded by L and the circle S of centre N and radius M. Since C lies within distance $M/2$ of O, and so $ON \leq M/2$, C lies inside or on S except for the arc PQ.

The segment PQ lies in $\bar{D}_0$ but the rest of L lies outside D_0. Also PQ divides D into (at most) two domains of which the domain D_1 containing O lies in D_0. If ω is the harmonic measure of the arc PQ of C w.r.t. D and ω_0, ω_1 are the harmonic measures of the segment PQ w.r.t. D_0 and D_1 respectively we see, by comparing values on the boundary of D_1, that

$$\omega(w) \leq \omega_1(w) \leq \omega_0(w) \qquad \text{in } D_1.$$

We apply this inequality at $w = O$. Using also Lemma 1 we obtain

$$\omega(O) \le \frac{PQ}{\pi}\left\{\frac{1}{ON} - \frac{ON}{M^2}\right\}.$$

If N lies on the chord PQ we have $ON \ge 1 - \delta$, where δ is the length of PQ and otherwise N lies outside D, so that $ON \ge 1$. Hence, if l is the length of the arc PQ of C, we deduce that

$$\omega(O) \le \frac{l}{\pi}\left\{\frac{1}{1-\delta} - \frac{1-\delta}{M^2}\right\} \le \frac{l(1+\epsilon)}{\pi}\left\{1 - \frac{1}{M^2}\right\}$$

if ϵ is a preassigned positive number and δ is sufficiently small depending on ϵ. By addition we deduce that, if γ is any arc of C having length l and harmonic measure $\omega(O)$ at the origin, then

$$\omega(O) \le \frac{l(1+\epsilon)}{\pi}\left\{1 - \frac{1}{M^2}\right\} \qquad \text{and so } \omega(O) \le \frac{l}{\pi}(1 - \frac{1}{M^2}),$$

since ϵ can be arbitrarily small.

We now denote by $\omega(z)$ the harmonic measure at the origin of an arc $[z_0, z]$ of C, where z_0 is a fixed point of C. Now if $u \in \mathcal{H}^+(C)$ we have

$$\int_C u(z)|dz| \ge \frac{\pi M^2}{M^2-1}\int_C u(z)d\omega(z) = \frac{\pi M^2}{M^2-1}u(0) = \frac{M^2}{2(M^2-1)}\int_{C_0} u(z)|dz|.$$

Using Theorem 4 we deduce that (1) with $K = 2(1 - M^{-2})$ holds whenever $u \in \mathcal{S}^+(C)$.

To complete the proof of Theorem 1 we calculate $K(\mathcal{S}^+, C)$ when C is a circle on the points $-r$, R as diameter, where $1 < R < r$. The function

$$w = \frac{z}{a+bz}, \text{ where } a = \frac{1}{2}\left\{\frac{1}{R} - \frac{1}{r}\right\}, b = \frac{1}{2}\left\{\frac{1}{R} + \frac{1}{r}\right\},$$

maps $|z| < 1$ onto the interior D of C. Also for $|z| = 1$

$$\left|\frac{dw}{dz}\right| = \frac{|a|}{|a+bz|^2} \ge \frac{a}{(a+b)^2} = \frac{R(r-R)}{2r},$$

with equality when $z = 1$. Thus by Theorem 3

$$K(\mathcal{S}^+, C) = \frac{2r}{R(r-R)}.$$

Choosing r large and R close to 1 we can achieve $K(\mathcal{S}^+, C) > 2 - \epsilon$. This proves Theorem 1.

5 Proof of Theorem 2

We proceed to calculate $K(\mathcal{A}, C)$, when C is a circle. Suppose first that C contains $|z| \leq 1$ in its interior and that $P(z)$ is regular in D and continuous in $\bar{D}$. We define

$$F(z) = \int_0^z P(\zeta)d\zeta$$

so that $F(z)$ is continuous in $\bar{D}$ and $F'(z) = P(z)$ in D. We choose a bilinear map

$$z = l(w) \tag{11}$$

which maps the circles $|w| = 1, R$ onto the circles C_O, C respectively, where $R > 1$. Then R is the module of the doubly connected domain bounded by C_0 and C. We define $f(w) = F[l(w)]$. Then

$$\int_{|w|=1} |f'(w)||dw| = \int_{|w|=1} |F'[l(w)]| \left|\frac{dz}{dw}\right| |dw| = \int_{|z|=1} |F'(z)||dz|$$

and

$$\int_{|w|=R} |f'(w)||dw| = \int_C |F'(z)||dz|$$

similarly. But since $u(w) = |f'(w)|$ is s.h. we have [6, p.64]

$$\int_{|w|=1} |f'(w)||dw| = \int_0^{2\pi} |f'(e^{i\theta})|d\theta \leq \int_0^{2\pi} |f'(Re^{i\theta})|d\theta = \frac{1}{R}\int_{|w|=R} |f'(w)||dw|.$$

This proves (1) with $K = 1/R$ as required. We note that equality holds if and only if $|f'(w)|$ is harmonic. Then $|f'(w)|$ is constant, $f(w) = aw + b$, $F(l(w)) = aw + b$. Then $F(z) = al^{-1}(z) + b$ maps C_0, C onto concentric circles. Thus $K(\mathcal{A}, C) = 1/R$ in this case.

Suppose next that C touches C_0. Let r_0 be the radius of C and let C_r be a circle concentric with C and of radius r, where $r > r_0$. The module of the region bounded by C_0 and C_r tends to 1 as $r \to r_0$. Thus by what we proved above we can, given $\epsilon > 0$, choose r and a polynomial $P(z)$ such that

$$\int_{C_0} |P(z)||dz| > (1-\epsilon)\int_{C_r} |P(z)||dz| \geq (1-\epsilon)\int_C |P(z)||dz|.$$

Thus $K(\mathcal{A}, C) > 1 - \epsilon$ and so $K(\mathcal{A}, C) \geq 1$. On the other hand we have for every polynomial P

$$\int_{C_0} |P(z)||dz| \leq \int_{C_r} |P(z)||dz|$$

by what we have already proved. Letting r tend to r_0 we obtain

$$\int_{C_0} |P(z)||dz| \leq \int_C |P(z)||dz|.$$

Thus $K(\mathcal{A}, C) \leq 1$ and so $K(\mathcal{A}, C) = 1$. This completes the proof of Theorem 2.

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The selection of mesh size for computation of waves over long times

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Dedicated to Jaap Korevaar on the occasion of his 70th birthday

Abstract

We examine numerical errors in finite-difference schemes for a linear wave-propagation problem from the point of view of how the propagation time influences the selection of the mesh size. We find that for a difference scheme with order of accuracy p and a long propagation time t, the mesh size should be scaled proportional to $t^{-1/p}$.

It is well known [5] that finite-difference approximations to wave equations introduce numerical errors. This effect has usually been studied in terms of the behavior at fixed location and time of the numerical solution as the mesh size tends to zero. The question we address here derives from the conflict between the facts that the numerical errors increase with distance, while phase information is relevant modulo 2π. We therefore address the following question. If one wants to compute a wave front over longer times or distances, by what ratio should the mesh size be decreased in order to maintain the same degree of accuracy?

We confine our attention to finite-difference methods for the uni-directional wave equation

$$\partial_t u + \partial_x u = 0 \tag{1}$$

on the set $-\infty < x < \infty$, $t > 0$ with initial data $u(x,0) = f(x)$. It is easily seen by substitution that the solution of (1) is

$$u = f(x - t). \tag{2}$$

In physical applications involving wave propagation one is interested in hyperbolic systems, not just the simple equation (1). The significance of equation (1) is that it may be viewed as the projection of a hyperbolic system $\partial_t u = A\,\partial_x u$ onto the subspace defined by a characteristic. Thus, our argument applies just as well to numerical methods for the acoustic wave equation, Maxwell's equation, or the equations of linear elasticity in 1 spatial dimension.

Because the wave speed for solutions of (1) is 1, the distance a wave propagates is equal to the travel time. In physical applications one usually thinks in terms of propagation distance, but our analysis is carried out in terms of travel time because this is more convenient for the analysis. Our principal result is that if the difference scheme has order of accuracy p, then for long propagation times t the mesh size h should be taken proportional to $1/t^{1/p}$. Consequently, it is wise to use difference methods of high accuracy (large p) when computing wave propagation over long times. At the end of the paper we illustrate our result with a computational example. Our result on the mesh size is asymptotic in nature, requiring the propagation time (and distance) to be large. The object of our example is to provide some insight into how large the time must be before the asymptotic result is valid. We find that for the Lax-Wendroff finite-difference method the asymptotic estimate is quite good at times as short as those needed to travel a distance of only 2 wave-lengths.

It is possible to give several heuristic arguments for our result. The simplest of these is as follows. The statement that the order of accuracy is p is equivalent to saying that at each time step the numerical error is of a size $O(h^{p+1})$. If there is no cancellation of errors, then after n time steps the accumulated error will be $O(nh^{p+1})$. For stability reasons the time step Δt is related to the spatial step h by

$$\Delta t = \lambda h \tag{3}$$

with constant λ. Therefore, the total time t is given by

$$t = n\Delta t = nh\lambda. \tag{4}$$

That is, the total error at time t should be $O(th^p)$, and this error will be constant if $h \sim t^{-1/p}$. We show that this result is correct, without the assumption of non-cancellation of errors.

Another heuristic argument for our result is based on the group velocity of waves in finite-difference grids. (Group velocity was very effectively used in [5] to study other properties of difference schemes.) The flaw in this approach is that the concept of group velocity is derived from an asymptotic analysis based on the method of stationary phase. The method of stationary phase is appropriate for nondissipative difference schemes, in which case the stationary points are also saddle points. The difficulty with this type of argument is that the method of stationary phase breaks down when saddle points coalesce, and we shall see that for $p > 1$ a wave front produces such a coalescence.

Background on numerical analysis. The principal tool for the analysis of finite-difference schemes for wave-propagation problems is the Fourier integral representation [4]. It is known that the appropriate tool for uniform asymptotic analysis involving a coalescence of saddle points uses a mapping to a generalized Airy integral. We therefore begin our discussion with a review of the connection between our problem and Airy integrals.

For the difference equation we introduce a uniform mesh with spatial step size h and time step Δt related by (3). As is customary in computational wave propagation, we let h and Δt vary, while keeping λ fixed. The spatial grid points are therefore $x_j = jh$, with $j = 0, \pm 1, \pm 2, \ldots$, and the temporal grid points are $t_n = n\Delta t$ with $n = 0, 1, 2, \ldots$. Let

u_n denote the approximation to u at time t_n. We consider difference schemes for (1) of the form

$$u_{n+1} = \sum_{j=-\infty}^{\infty} c_j T^j u_n, \tag{5}$$

where T is the translation operator $Tu_n(x) = u_n(x+h)$. For the sake of convenience, we restrict our attention to explicit difference schemes, so that only a finite number of coefficients c_j in the sum (5) is nonzero. We require the coefficients c_j to be real constants. The c_j could depend on the mesh size h, but this would not normally occur in finite-difference approximations to (1). We take the natural initial condition for (5), that $u_0 = f$ at points on the spatial grid. Note that the difference scheme (5) need not be restricted to the grid $x_j = jh$, but that it makes sense on the entire real line. This point is important in our discussion of the effect of varying the mesh size.

Our analysis of the difference scheme (5) is based on the Fourier transform

$$\widetilde{u}_n(\xi) = \int_{-\infty}^{\infty} u_n(x) \exp\{-ix\xi\}\, dx \tag{6}$$

with inverse Fourier transform

$$u_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{u}_n(\xi) \exp\{ix\xi\}\, d\xi. \tag{7}$$

We assume that these integrals make sense, that is, that the integrands are Lebesgue integrable. Because the Fourier transform of Tu is $e^{ih\xi}\widetilde{u}(\xi)$, an application of (6) to the finite-difference equation (5) shows that if u_n has a Fourier transform, then

$$\widetilde{u}_{n+1}(\xi) = F(h\xi)\widetilde{u}_n(\xi),$$

where

$$F(\xi) = \sum c_j \exp\{ij\xi\}. \tag{8}$$

The function F is the symbol of the difference scheme, and for explicit schemes (5) F is a trigonometric polynomial. From (8) it follows that

$$\widetilde{u}_n(\xi) = F(h\xi)^n \widetilde{u}_0(\xi). \tag{9}$$

We need to define the order of accuracy of the difference scheme (5). This in turn is defined in terms of the local truncation error, which is the amount by which a smooth solution of (1) fails to satisfy (5). The first step in obtaining the local truncation error is a formal expansion of a solution u of (1) into Taylor series

$$T^j u = \sum_{k=0}^{\infty} \frac{(jh)^k}{k!}\, \partial_x^k u. \tag{10}$$

The right-hand side of (5) therefore takes the form

$$\sum_k \left(\sum_j (jh)^k c_j \right) \frac{\partial_x^k u}{k!}. \tag{11}$$

Here and in the rest of this paper, we take $(jh)^k = 1$ when $j = 0$ and $k = 0$.

For the left-hand side of (5) we use the Taylor series

$$u(x, t + \lambda h) = \sum_k \frac{(\lambda h)^k}{k!} \partial_t^k u. \tag{12}$$

If u is a smooth solution of (1), we may differentiate (1) repeatedly to obtain

$$\partial_t^k u = (-1)^k \partial_x^k u. \tag{13}$$

Upon combining (12–13), we find that

$$u(x, t + \lambda h) = \sum_k \frac{(-\lambda h)^k \partial_x^k u}{k!}. \tag{14}$$

The order of accuracy of the difference scheme (5) is determined by comparing consecutive terms of the series (11) and (14).

Definition. Let p be a positive integer. The difference scheme (5) is said to approximate (1) with *order of accuracy* p if the terms with $k = 0, \ldots, p$ in equations (11) and (14) are equal and the terms with $k = p + 1$ are not equal. That is, order of accuracy p is equivalent to the two conditions

$$\sum_j j^k c_j = (-\lambda)^k, \qquad k = 0, \ldots, p, \tag{15}$$

and

$$\sum_j j^{p+1} c_j \neq (-\lambda)^{p+1}. \tag{16}$$

The connection between the symbol F and the order of accuracy is contained in the following lemma.

Lemma 1. *If the difference scheme* (5) *approximates* (1) *with order of accuracy* p, *then there exists a nonzero coefficient* a_{p+1} *such that the Maclaurin series of* $\log F$ *takes the form*

$$\log F(\xi) = -i\lambda\xi + a_{p+1}\xi^{p+1} + O(|\xi|^{p+2}). \tag{17}$$

Remark. There would be no numerical errors if we had $\log F(\xi) = -i\lambda\xi$.

Proof. This is classical numerical analysis, see Richtmyer and Morton [4, p. 68], but we include a proof for the sake of completeness. By (8) the Maclaurin expansion of $F(\xi)$ is

$$F(\xi) = \sum_k \frac{(i\xi)^k}{k!} \sum_j j^k c_j.$$

Note that the relation (15) with $k = 0$ implies that $F(0) = 1$. It now follows from (15–16) that

$$F(\xi) = \sum_{k=0}^{p+1} \frac{(-i\lambda\xi)^k}{k!} + a_{p+1}\xi^{p+1} + O(|\xi|^{p+2}) \tag{18}$$

with $a_{p+1} \neq 0$. The lemma now follows upon comparing this series with the Maclaurin series for the exponential of the right-hand side of (17). Observe that the sum in (18) is taken up to $k = p+1$ and that a_{p+1} in (18) is identical to a_{p+1} in (17).

Wave fronts and uniform asymptotics. We begin our analysis by writing the solution u_n to (5) as the inverse Fourier transform to (9),

$$u_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\xi)\exp\{n\log F(h\xi) + ix\xi\}\, d\xi. \tag{19}$$

Let us consider the various parameters. In most studies of convergence of difference schemes one keeps t fixed, with the constraint (4) and with $h \to 0$ and $n \to \infty$. We maintain (4), but we let x and t go to infinity along rays

$$x = \omega t.$$

The parameter ω represents a propagation speed in the grid. In terms of this notation we find that

$$u_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\xi)\exp\{t\Phi(\xi,\omega,h)\}\, d\xi \tag{20}$$

with

$$\Phi(\xi,\omega,h) = \frac{1}{h\lambda}\log F(h\xi) + i\omega\xi. \tag{21}$$

It follows from (17) that the first few terms of the Maclaurin series of Φ are

$$\Phi(\xi,\omega,h) = i(\omega-1)\xi + \frac{a_{p+1}h^p}{\lambda}\xi^{p+1} + \frac{a_{p+2}h^{p+1}}{\lambda}\xi^{p+2} + O(h^{p+2}|\xi|^{p+3}). \tag{22}$$

A saddle-point analysis of the asymptotic behavior as $t \to \infty$ of the integral (20) would start with the solution of the equation

$$\partial_\xi\Phi(\xi,\omega,h) = 0$$

to determine the locations of the saddle points ξ. It is clear from (22) that $\partial_\xi\Phi$ has a zero of order p at $\xi = 0$ when $\omega = 1$. But $\omega = 1$ corresponds to propagation at the speed 1, which is the speed of most interest for solutions of (5). Thus, the analysis of the asymptotic behavior near a wave front of the integral (20) as $t \to \infty$ involves the coalescence of p saddle points at $\xi = 0$. This coalescence of saddle points implies that we cannot use an argument based on group velocity to analyze behavior of (20) as $t \to \infty$.

We remark that on the basis of (20) and (22) one might expect to have the approximation

$$u_n(x) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\xi) \exp\left\{ i(x-t)\xi + \frac{a_{p+1} t h^p}{\lambda} \xi^{p+1} \right\} d\xi. \tag{23}$$

This gives yet another heuristic argument for saying that the numerical error for large values of t is $O(th^p)$. Note also that the left-hand side of (23) is the solution of the partial differential equation

$$\partial_t u = -\partial_x u + \frac{a_{p+1} h^p}{i^{p+1} \lambda} \partial_x^{p+1} u \tag{24}$$

with initial data $u(x,0) = f(x)$. The approximation (23) is a heuristic widely used in numerical analysis in the equivalent formulation (24), and it is called the modified equation [6]. It happens, however, that (23) is valid only under very restricted conditions on the size of $|x-t|$ because of the influence of the neglected terms in the series (22) when $th^{p+1}|\xi|^{p+2}$ is not small.

What is needed to analyze the asymptotic behavior of the integral (20) as $t \to \infty$ and $h \to 0$ is a method which is uniform with respect to coalescence of saddle points. Such a uniform asymptotics may be done, and it is based on the mapping

$$\xi = \psi(\zeta, t, \omega, h) \tag{25}$$

from the Weierstrass preparation theorem [3, pp. 144–146] of $t\Phi$ to a polynomial

$$\Psi(\zeta, t, \omega, h) = \frac{\zeta^{p+1}}{p+1} + \sum_{k=0}^{p-1} \gamma_k(t, \omega, h) \zeta^k.$$

To be specific, the mapping (25) is bijective in a neighborhood of $(\xi, \omega) = (0, 1)$ and takes $t\Phi$ to the canonical form

$$t\Phi(\xi, \omega, h) = \Psi(\zeta, t, \omega, h). \tag{26}$$

The asymptotic behavior of (20) as $t \to \infty$ is obtained in terms of a sequence of polynomials P_m of degree $p-1$ in ζ obtained as remainders in a division processes. The initial step is

$$\tilde{f}(\psi(\zeta, t, \omega, h))\, \partial_\zeta \psi = P_0 - Q_1\, \partial_\zeta \Psi.$$

Subsequent remainders P_m are obtained by dividing the derivative of the quotient Q_m by $\partial_\zeta \Psi$

$$\partial_\zeta Q_m = P_m - Q_{m+1}\, \partial_\zeta \Psi \tag{27}$$

We are now ready to state a lemma on the asymptotic behavior of (20).

Lemma 2. *The asymptotic behavior of the integral* (20) *as* $t \to \infty$ *which is uniform with respect to* h *in in a neighborhood of* 0 *and* $\omega = x/t$ *in a neighborhood of* 1 *is given by the formal expansion*

$$u_n(x) \sim \frac{1}{2\pi} \sum_{m=0}^{\infty} \int_{\Gamma} P_m \exp\{\Psi\}\, d\zeta.$$

Here, P_m *is defined by* (27) *and* Γ *is a path in the complex plane determined by the mapping* (25).

Proof. This result is classical asymptotic analysis, and a proof may be found, for example, in [2, pp. 457–458]. Formally the asymptotic expansion is obtained by making the change of variables (25–26) and sucessively performing the division (27) and integrating by parts.

We remark that Estep *et al.* [1] have also applied Lemma 2, but to a different problem in numerical analysis. We now apply Lemma 2 to prove the following theorem.

Theorem. *Suppose that the difference scheme* (5) *is stable and approximates* (1) *with order of accuracy* p *for some positive integer* p. *Suppose also that the temporal mesh size* Δt *is tied to the spatial mesh size* h *by the relation* $\Delta t = \lambda h$ *for some positive constant* λ. *Then in order to maintain an error bound*

$$\sup_x |u_n(x) - u(x,t)| \leq E$$

with $t = n\Delta t$, *one should select the mesh size* h *according to*

$$h = \frac{\text{const}}{t^{1/p}}.$$

Proof. Let us define a parameter β by the relation

$$\beta = \frac{h^p t}{\lambda}. \tag{28}$$

Note that the inclusion of the mesh ratio $\lambda = \Delta t/h$ in the definition of β simplifies some of the formulas which appear later. By Lemma 2 the proof of the theorem is reduced to investigations the coefficients γ_k and the mapping defined by (25). In particular, we show that for large values of the time t, the coefficients γ_k, and the mapping (25) depend to first order only on the parameter $\beta = h^p t/\lambda$ and on the distance to the wave front $x - t$.

The cases $p = 1$, $p = 2$, and $p > 2$ are different, so we treat them separately, starting with $p = 1$. With the notation (28) it follows from (22) that the mapping (25) takes the form

$$t\Phi(\xi, x/t, h) = i(x-t)\xi + \beta a_2 \xi^2 + O(h|\xi|^3) = \frac{\zeta^2}{2} + \gamma_0. \tag{29}$$

That is, for $p = 1$ the canonical form is a Gaussian instead of an Airy function. It is clear that as $h \to 0$ the function $\partial_\xi \Phi(\xi, x/t, h)$ has a zero ξ_0 given by

$$\xi_0 = \frac{i(t-x)}{2\beta a_2} + O(h). \tag{30}$$

The value of γ_0 such that $\xi = \xi_0$ is mapped by (29) onto $\zeta = 0$ is therefore given by

$$\gamma_0 = \frac{(x-t)^2}{4\beta a_2} + O(h) \tag{31}$$

as $h \to 0$, and the mapping (29) takes the form

$$\zeta = \sqrt{2\beta a_2}(\xi - \xi_0). \tag{32}$$

For $p = 1$ it is clear from (30) that to first order in h the mapping (32) defined by (25–26) depends only on β and $x - t$. Furthermore, we see from (31) that the same is true of γ_0. This proves the theorem for the case $p = 1$.

For $p > 2$ we perform the first step in the construction of a special case of the Weierstrass mapping. Specifically, we show that for $p > 2$ the relation (26) as $h \to 0$ may be written in the form

$$\zeta = \kappa\xi + \mu h\xi^2 + O(h^2) \tag{33}$$

with

$$\kappa = \left((p+1)\beta a_{p+1}\right)^{1/(p+1)} \tag{34}$$

and

$$\mu = \frac{\beta a_{p+2}}{(p+1)\kappa^p}. \tag{35}$$

In fact, a use of (33) to replace ζ in the right-hand side of (26) leads to

$$t\Phi(\xi, x/t, h) = \frac{\kappa^{p+1}\xi^{p+1}}{p+1} + h\kappa^p\mu\xi^{p+2} + \sum_{k=0}^{p-1}\gamma_k\{\kappa^k\xi^k + hk\mu\kappa^{k-1}\xi^{k+1}\} + O(h^2). \tag{36}$$

We equate terms of the expansion (36) with the corresponding terms from the left-hand side of (22),

$$t\Phi(\xi, x/t, h) = i(x-t)\xi + \beta a_{p+1}\xi^{p+1} + \beta a_{p+2}h\xi^{p+2} + O(h^2|\xi|^{p+3}). \tag{37}$$

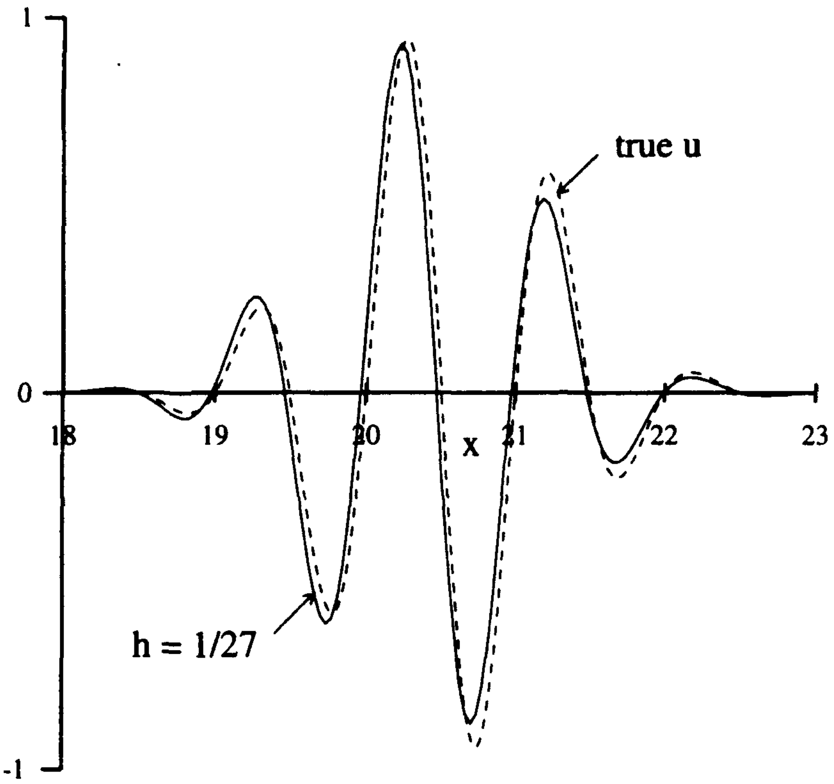
The ξ^{p+1}-terms give the value (34) of κ, and the $h\xi^{p+2}$-terms give the value (35) of μ. From the lower-order terms we find that

$$\gamma_0 = 0, \qquad \gamma_1 = \frac{i(x-t)}{\kappa}, \qquad \gamma_2 = -\frac{\mu\gamma_1}{\kappa^2}, \tag{38}$$

and $\gamma_k = 0$ for $k = 3, \ldots, p-1$. Again we see that up to terms of order h, the mapping (33) and the coefficients γ_k depend only on the parameter β and the distance $x - t$ to the wave front. This proves the theorem for the case $p > 2$.

The case $p = 2$ differs from $p > 2$ in that in place of (33) we use

$$\zeta = \delta h + \kappa\xi + \mu h\xi^2 + O(h^2).$$



hscale=200 vscale=200 hoffset=-270 voffset=400 angle=270

Fig. 1. Wave pulse at time $t = 18$.

Comparison of the terms of the power series of both sides of (26) again yields the values (34) and (35) (with $p = 2$), as well as the value (38) for γ_1. What is different is that the terms in $h\xi^2$ and the constant terms give the values

$$\delta = -\frac{\gamma_1 \mu}{\kappa^2} \quad \text{and} \quad \gamma_0 = -\gamma_1 \delta h.$$

Once again, we find that for $p = 2$ the mapping and the coefficients γ_k depend up to order h only on the parameter $\beta = h^p t/\lambda$ and the distance $x - t$ to the wave front. This completes the proof of the theorem.

Numerical example. We illustrate the theorem by some computations with the Lax-Wendroff scheme [4, p. 302]

$$u_{n+1} = \frac{1}{2}\lambda(1+\lambda)T^{-1}u_n + (1-\lambda^2)u_n - \frac{1}{2}\lambda(1-\lambda)Tu_n. \tag{39}$$

The intial data is

$$u(x,0) = f(x) = \exp\left\{-(x-x_0)^2\right\}\sin\{2\pi x\}, \tag{40}$$

This choice of initial data produces an oscillatory pulse, with the oscillations having period 1, and this period serves as a natural time scale. The solution is a translated pulse as shown

t	h	Error
2	1/27	0.02539
8	1/54	0.02552
18	1/81	0.02554

Table 1. Dependence of accuracy on mesh size and distance.

in Fig. 1 at time $t = 18$. The computations were performed on a bounded interval $0 \leq x \leq X = 25$, and we set $x_0 = 2.5$ in (40). For the boundary condition at $x = 0$ we set $u = 0$ for both the partial differential equation (1) and the difference scheme (39). The difference scheme (39) also requires a boundary condition at the right-hand boundary $x = X$. We rather arbitrarily chose $u_n(X) = 0$ and stopped the calculation before the oscillatory pulse reached $x = X$. We chose the value $\lambda = 0.9$ because the stability requirement for (39) is that $0 < \lambda \leq 1$.

It is known that the order of accuracy of (39) is $p = 2$ [4]. According to the theorem, the numerical error for $t \gg 1$ should remain nearly constant if $ht^{1/2}$ is kept constant. This effect is illustrated in Table 1, where each case has $ht^{1/2} = \sqrt{2}/27$. In this table the column labelled 'Error' is an approximation by the trapezoid rule of the relative error in the L_2-norm

$$\frac{\|u_n - u\|}{\|u\|},$$

where the norm is taken at fixed time

$$\|u\|^2 = \int_0^X |u(x,t)|^2 \, dx.$$

In Fig. 1 we display only the true solution u at time $t = 18$ and the numerical solution with spatial step size $h = 1/27$. The numerical solutions with step sizes $h = 1/54$ and $h = 1/81$ are not shown, because they are nearly indistinguishable from u.

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Uniform multi-parameter limit transitions in the Askey tableau

Tom H. Koornwinder

Dedicated to prof. Jaap Korevaar

1. Elementary limit formulas

We consider the classical orthogonal polynomials as monic polynomials $p_n(x) = x^n +$ terms of degree less than n. We have

- Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ with respect to weight function $(1-x)^\alpha(1+x)^\beta$ on $(-1,1)$;
- Laguerre polynomials $\ell_n^\alpha(x)$ with respect to weight function $e^{-x}x^\alpha$ on $(0,\infty)$;
- Hermite polynomials $h_n(x)$ with respect to weight function e^{-x^2} on $(-\infty,\infty)$.

Note that, for $\alpha\to\infty$, the rescaled Jacobi weight function $(1-x^2/\alpha)^\alpha$ on $(-\alpha^{\frac12},\alpha^{\frac12})$ tends to the Hermite weight function e^{-x^2} on $(-\infty,\infty)$. Accordingly we have the limit formula

$$\lim_{\alpha\to\infty}\alpha^{n/2}p_n^{(\alpha,\alpha)}(x/\alpha^{\frac12}) = h_n(x).$$

Also, for $\beta\to\infty$, the rescaled Jacobi weight function $x^\alpha(1-x/\beta)^\beta$ on $(0,\beta)$ tends to the Laguerre weight function $x^\alpha e^{-x}$ on $(0,\infty)$. Accordingly we have the limit formula

$$\lim_{\beta\to\infty}(-\beta/2)^n\,p_n^{(\alpha,\beta)}(1-2x/\beta) = \ell_n^\alpha(x).$$

This can be graphically indicated in the (α,β)-parameter plane extended with the lines $\{(\alpha,\beta)\mid \alpha=\infty,\ -1<\beta\le\infty\}$ and $\{(\alpha,\beta)\mid -1<\alpha\le\infty,\ \beta=\infty\}$. When we start with a point (α,α) then we can draw a diagonal arrow to the (Hermite) point (∞,∞) and a vertical arrow to the (Laguerre) point (α,∞).

The celebrated Favard theorem states that $\{p_n\}_{n=0,1,2,\ldots}$ is a system of monic orthogonal polynomials with respect to a positive orthogonality measure if and only if a recurrence relation

$$\begin{aligned} x\,p_n(x) &= p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x), \quad n=1,2,\ldots, \\ x\,p_0(x) &= p_1(x) + B_0\,p_0(x), \\ p_0(x) &= 1, \end{aligned} \tag{1.1}$$

is valid with $C_n>0$ and B_n real. Below, when we will give this recurrence relation with explicit coefficients then we will silently assume that the case $n=0$ has the same analytic form as the case $n>0$, but with the term $C_n\,p_{n-1}(x)$ omitted.

If the coefficients B_n and C_n are given then p_n is completely determined by this recurrence relation. In particular, if B_n and C_n would continuously depend on some parameter

λ then $p_n(x)$ will also continuously depend on λ. For example, Hermite polynomials satisfy the recurrence relation

$$x\,h_n(x) = h_{n+1}(x) + \tfrac{1}{2}n\,h_{n-1}(x). \tag{1.2}$$

Now consider rescaled Laguerre polynomials

$$p_n(x) = p_n(x;\alpha,\rho,\sigma) := \rho^n\,\ell_n^\alpha(\rho^{-1}x - \sigma).$$

From the well-known recurrence relation for Laguerre polynomials we find for these rescaled polynomials:

$$x\,p_n(x) = p_{n+1}(x) - \rho\,(2n+\alpha+1+\sigma)\,p_n(x) + \rho^2\,n\,(n+\alpha)\,p_{n-1}(x). \tag{1.3}$$

We would like to make the rescaling in such a way that, as $\alpha \to \infty$, $p_n(x)$ will tend to $h_n(x)$. It is easy to see how to do this when we compare (1.2) and (1.3). Put $\rho := (2\alpha)^{-\frac{1}{2}}$, $\sigma := -\alpha$. Then (1.3) becomes

$$p_{n+1}(x) - (2\alpha)^{-\frac{1}{2}}\,(2n+1)\,p_n(x) + \frac{n\,(n+\alpha)}{2\alpha}\,p_{n-1}(x).$$

The recurrence coefficients now tend to 0 resp. $\frac{1}{2}n$ as $\alpha \to \infty$. Hence $p_n(x) \to h_n(x)$ as $\alpha \to \infty$, i.e.,

$$\lim_{\alpha\to\infty} (2\alpha)^{-\frac{1}{2}n}\,\ell_n^\alpha((2\alpha)^{\frac{1}{2}}x + \alpha) = h_n(x).$$

Thus, in the extended (α,β)-parameter plane we can also start at a Laguerre point (α,∞) and draw a horizontal arrow to the Hermite point (∞,∞).

2. Uniform limit of Jacobi polynomials

It is now natural to conjecture that we might also make these limit transitions in the parameter plane in a more uniform way, i.e., to make such a rescaling of the Jacobi polynomials that they depend continuously on (α,β) in the extended parameter plane and reduce to (possibly rescaled) Laguerre and Hermite polynomials on the boundary lines and boundary vertex at infinity, respectively. For this purpose we consider Jacobi polynomials with arbitrary rescaling:

$$p_n(x) := \rho^n\,p_n^{(\alpha,\beta)}(\rho^{-1}x - \sigma). \tag{2.1}$$

These polynomials satisfy recurrence relations (1.1) with

$$\begin{aligned} C_n :=& \rho^2\,\frac{4n\,(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)}{(2n+\alpha+\beta-1)\,(2n+\alpha+\beta)^2\,(2n+\alpha+\beta+1)} = \frac{\rho^2\,\alpha\beta\,(\alpha+\beta)}{(\alpha+\beta)^4} \\ &\times \frac{4n\,(1+n/\alpha)\,(1+n/\beta)\,(1+n/(\alpha+\beta))}{(1+(2n-1)/(\alpha+\beta))\,(1+2n/(\alpha+\beta))^2\,(1+(2n+1)/(\alpha+\beta))} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} B_n :=& \rho\left(\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)} + \sigma\right) \\ =& \rho\left(\frac{\beta-\alpha}{\beta+\alpha}\,\frac{1}{1+2n/(\alpha+\beta)}\,\frac{1}{1+(2n+2)/(\alpha+\beta)} + \sigma\right). \end{aligned} \tag{2.3}$$

From (2.2) we see that the choice

$$\rho := \frac{(\alpha+\beta)^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}\,\beta^{\frac{1}{2}}} \tag{2.4}$$

makes C_n continuous on (α,β) in the extended parameter plane. Next we see from (2.3) that the choice

$$\sigma := \frac{\alpha-\beta}{\alpha+\beta} \tag{2.5}$$

makes B_n continuous in (α,β) (extended) as well. Indeed, we can now rewrite

$$B_n = \frac{\beta^{-1}-\alpha^{-1}}{(\alpha^{-1}+\beta^{-1})^{\frac{1}{2}}}\,\frac{4n+2+4n\,(n+1)/(\alpha+\beta)}{(1+2n/(\alpha+\beta))\,(1+(2n+2)/(\alpha+\beta))},$$

which is continuous in (α^{-1},β^{-1}) for $\alpha^{-1},\beta^{-1}\geq 0$.

As a result we can consider the (α^{-1},β^{-1})-parameter plane. For $\alpha^{-1},\beta^{-1}>0$ we have the rescaled Jacobi polynomials (2.1) with ρ and σ given by (2.4) and (2.5). These polynomials extend continously to the closure $\{(\alpha^{-1},\beta^{-1})\mid \alpha^{-1},\beta^{-1}\geq 0\}$. On the boundary lines $\{(\alpha^{-1},0)\mid \alpha^{-1}>0\}$ and $\{(0,\beta^{-1})\mid \beta^{-1}>0\}$ these polynomials become rescaled Laguerre polynomials. On the boundary vertex $(0,0)$ the polynomials become Hermite polynomials.

3. Uniform limits in the Askey tableau

The Askey tableau is given by the following chart (cf. Askey & Wilson [1, Appendix].

$$\begin{array}{ccccccccc}
\text{Wilson} & & & & & & \text{Racah} & & \\
\downarrow & \searrow & & & & \swarrow & & \searrow & \\
\text{cont. dual Hahn} & & \text{cont. Hahn} & & \text{Hahn} & & & & \text{dual Hahn} \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \searrow & & \swarrow & \downarrow \\
 & & & & & \swarrow & & \searrow & \\
\text{Meixner-Pollaczek} & & \text{Jacobi} & & \text{Meixner} & & & & \text{Krawtchouk} \\
 & \searrow & \downarrow & \swarrow & & \searrow & & \swarrow & \\
 & & \text{Laguerre} & & & & \text{Charlier} & & \\
 & & & \searrow & & \swarrow & & & \\
 & & & & \text{Hermite} & & & &
\end{array}$$

The various families of orthogonal polynomials mentioned here are all of classical type, i.e., the orthogonal polynomials $\{p_n\}_{n=0,1,\ldots}$ satisfy an equation of the form

$$L\,p_n = \lambda_n\,p_n,$$

where L is some second operator (differential or difference) which does not depend on n. The arrows in the chart mean limit transitions between the various families. The number of additional parameters on which the polynomials depend, decreases as we go further down in the chart. In the top row there are 4 parameters. In each subsequent row there is one parameter less. The Hermite polynomials in the bottom row no longer depend on parameters. The families in the left part of the chart consist of polynomials being orthogonal with respect to an absolutely continuous measure, while the ones in the right part are orthogonal with respect to a discrete measure. In the case of Racah, Hahn, dual Hahn and Krawtchouk polynomials the support of the measure has finite cardinality, say $N+1$, and we consider only polynomials up to degree N.

All the polynomials in this chart have explicit expressions as hypergeometric functions. For instance, *Jacobi polynomials* are given by

$$P_n^{(\alpha,\beta)}(x) = \text{const. } {}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1 \\ \alpha+1\end{matrix}; x\right].$$

Hahn polynomials are given by

$$Q_n(x;\alpha,\beta,N) := {}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right], \quad n = 0,1,\ldots,N,$$

while they satisfy orthogonality relations

$$\sum_{x=0}^{N} Q_n(x)\, Q_m(x)\, \frac{(\alpha+1)_x\, (\beta+1)_{N-x}}{x!\,(N-x)!} = 0, \quad n \neq m.$$

Racah polynomials are given by

$$R_n(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = {}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right],$$

where $\gamma+1 = -N$ and $n = 0,1,\ldots,N$.

Now consider monic Racah polynomials

$$r_n(x;\alpha,\beta,-N-1,\delta) := \text{const. } R_n(x;\alpha,\beta,-N-1,\delta) = x^n + \cdots$$

and rescaled monic Racah polynomials

$$p_n(x) := \rho^n\, r_n(\rho^{-1}x - \sigma;\alpha,\beta,-N-1,\delta). \tag{3.1}$$

Then the $p_n(x)$ satisfy recurrence relations (1.1) with

$$\begin{aligned} C_n = \rho^2\, n\,(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)\,(n+\beta+\delta) \\ \times \frac{(\delta-\alpha-n)\,(n+N+\alpha+\beta+1)\,(N+1-n)}{(2n+\alpha+\beta-1)\,(2n+\alpha+\beta)^2\,(2n+\alpha+\beta+1)}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} B_n = \rho\Big(\frac{(n+\alpha+\beta+1)\,(n+\alpha+1)\,(n+\beta+\delta+1)\,(N-n)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)} \\ + \frac{n\,(n+\beta)\,(\delta-\alpha-n)\,(n+N+\alpha+\beta+1)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)} + \sigma\Big). \end{aligned} \tag{3.3}$$

Now we would like to express ρ and σ in such a way in terms of α, β, δ and N that the polynomials $p_n(x)$ in (3.1) continuously depend on these parameters up to boundaries at infinity in the four-parameter plane, and such that Hahn polynomials and all families which can be reached from the Hahn polynomials in the Askey tableau, are obtained as polynomials $p_n(x)$ with parameters on the boundary. This is a task analogous to what we did in section 2, but much more complicated. Again, formula (3.2) for C_n should suggest a choice for ρ and next formula (3.3) for B_n should lead to the choice of σ. Thus we arrive at

$$\rho := \left(\frac{(\alpha+\beta)^3}{\alpha\beta\,(\beta+\delta)\,(N+\alpha+\beta)\,(\delta-\alpha)\,N}\right)^{\frac{1}{2}}, \tag{3.4}$$

$$\sigma := -\frac{N\,(\alpha+1)\,(\beta+\delta+1)}{\alpha+\beta+2}. \tag{3.5}$$

Now turn from parameters α, β, δ, N to parameters α, b, d, ν by the substitution

$$\beta = b\alpha, \quad \delta = (bd\nu+1)\,\alpha, \quad N = b\,\nu. \tag{3.6}$$

Then we can prove the following theorem. Computations for this are somewhat tedious. Some of them I performed with the help of Maple V.

Theorem The rescaled monic Racah polynomials $p_n(x)$ given by (3.1), with (3.4), (3.5) and (3.6) being substituted, are continuous in $(\alpha^{-1}, b^{-1}, d^{-1}, \nu^{-1})$ for $\alpha^{-1}, b^{-1}, d^{-1}, \nu^{-1} \geq 0$. When we restrict to any of the lower dimensional boundaries than we obtain rescaled versions of other polynomials in the Askey tableau, as given below.

dimension	specialization	orthogonal polynomial family
4		Racah
3	$d = \infty$	Hahn
3	$\nu = \infty$	Jacobi
3	$b = \infty$	Meixner
3	$\alpha = \infty$	Krawtchouk
2	$d, \nu = \infty$	Jacobi
2	$d, b = \infty$	Meixner
2	$d, \alpha = \infty$	Krawtchouk
2	$\nu, b = \infty$	Laguerre
2	$\nu, \alpha = \infty$	Hermite
2	$b, \alpha = \infty$	Charlier
1	$d, \nu, b = \infty$	Laguerre
1	$d, b, \alpha = \infty$	Charlier
1	$d, \nu, \alpha = \infty$	Hermite
1	$\nu, b, \alpha = \infty$	Hermite
0	$\nu, b, d, \alpha = \infty$	Hermite

It is probably possible to formulate an anlogous theorem with Hahn polynomials being replaced by dual Hahn polynomials. We should start then with a different part of four-parameter space for the Racah polynomials. For Wilson polynomials we can start with three different regions in four-parameter space. For each of these three cases there are different limits in the Askey tableau (cf. [2, Table 4]). We can hope that for each of these three cases a result analogous to the above theorem will hold. A further possible extension might involve q as a fifth parameter. One might also try to include the limits to non-polynomial special functions like Bessel functions and Jacobi functions (cf. [2]).

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Living in a Faraday cage

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1 Introduction: two questions

It is nice to be able to tell you something about a problem area that a few of us here have been investigating. Some of the following has been joint work with J.L.H. Meyers. There is also ongoing work by and with A.B.J. Kuijlaars and M.A. Monterie.

We have studied two principal questions that have a lot to do with each other, especially in the case of the unit sphere $S = S(0,1)$ in $\mathbf{R}^3$, so let us focus on it. For large N, we consider N-tuples $Z_N = (\zeta_1, \ldots, \zeta_N)$ cf points on S.

Question 1.1 In what I call "the Faraday caage problem for discrete charges" on S, we place point charges $1/N$ at the N points of Z_N. We ask: How small can one make the resulting electrostatic field $\mathcal{E}(x, Z_N)$ on balls $B(0,r)$, $(r < 1)$ by judicious choice of Z_N? To get closer to physical reality, we would like to demand that our N-tuple of point charges have minimal potential energy

$$V(Z_N) = \frac{1}{N^2} \sum_{j,k=1, j\neq k}^{N} \frac{1}{|\zeta_j - \zeta_k|}. \tag{1.1}$$

However, for technical reasons we won't impose that restriction here.

For a continuous equilibrium distribution of total charge 1 on S, the field inside S would be zero and the potential would be constant (equal to 1) throughout the closed unit ball. In our case the field $\mathcal{E}(x) = \mathcal{E}(x, Z_N)$ would be minus the gradient of the potential

$$U(x) = U(x, Z_N) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\zeta_j - x}. \tag{1.2}$$

Question 1.2 In what is called the "Chebyshev-type quadrature problem" for S, we approximate the average of functions f over S,

$$\frac{1}{4\pi} \int_S f(\zeta) ds(\zeta) = \int_S f(\zeta) d\sigma(\zeta), \qquad (\sigma = s/4\pi)$$

by the arithmetic mean

$$\frac{1}{N}\{f(\zeta_1) + \cdots + f(\zeta_N)\}.$$

We now ask: How should one choose the N-tuple Z_N in order to get small quadrature remainder

$$R(f, Z_N) = \int_S f(\zeta) d\sigma(\zeta) - \frac{1}{N} \sum_{j=1}^{N} f(\zeta_j) \tag{1.3}$$

for a large class of functions f? We would call an N-tuple of nodes Z_N "good" if the corresponding quadrature remainder vanishes or is very small for all polynomials $f(x_1, x_2, x_3)$ up to relatively high degree p.

In the past the speaker and coauthors T. Geveci and R.A. Kortram have studied the analog of Question 1 for very smooth plane Jordan curves, cf. [5,2,6] and Section 3 below. More recently we have studied Chebyshev-type quadrature for a variety of simple surfaces [7,9].

2 Equivalence of the questions for the sphere

For the sphere S there is a close connection between the two questions: good N-tuples of nodes correspond to configurations of point charges for which the electrostatic field is very small inside S. For example, we have the following simple

Theorem 2.1 cf. [8] *For a given N-tuple Z_N on S, one has*

$$R(f, Z_N) = 0$$

for all polynomials f of degree $\leq p$ if and only if

$$\mathcal{E}(x, Z_N) = \mathcal{O}(|x|^p) \qquad \text{for } 0 \leq x \leq r_0 < 1.$$

There is a corresponding more general "Equivalence Theorem" involving very small remainders $R(f, Z_N)$ for all polynomials of degree $\leq p$ with $\sup_S |f| = 1$ [8]. Hence if for some p there is a polynomial f of degree p for which $R(f, Z_N)$ is not very small relative to $\sup_S |f|$, the field $\mathcal{E}(x, Z_N)$ can not be too small either. On the sphere, every polynomial f of degree $\leq p$ can be written as a linear combination of spherical harmonics of order $\leq p$. Counting linearly independent spherical harmonics, one finds that for given Z_N, there is always a non-negative polynomial f of degree $\leq 2\sqrt{N}$ with $\sup_S |f| = 1$ which vanishes on Z_N. Estimating $R(f, Z_N) = \int_S f$ and applying the Equivalence Theorem, one obtains

Theorem 2.2 cf. [8] *For every N-tuple Z_N of points on S,*

$$\sup_{|x|=r} |\mathcal{E}(x, Z_N)| > \frac{r^{2\sqrt{N}-1}}{4(\sqrt{N}+1)^3}, \qquad 0 < r < 1.$$

For large N, the lower bound is extremely small, but we believe that it can "almost" be achieved for special N-tuples. However, we are far from a proof for the corresponding

Conjecture 2.3 For $r_0 \in (0,1)$, there are constants A and $c > 0$ such that for special N-tuples $\tilde{Z}_N$ with $N \to \infty$,

$$\sup_{|x| \leq r} |\mathcal{E}(x, \tilde{Z}_N)| \leq A r^{c\sqrt{N}}, \qquad 0 \leq r \leq r_0.$$

3 The case of plane curves

A first argument in support of Conjecture 2.3 is provided by the known state of affairs in the case of the unit circle $C = C(0,1)$ in the plane $\mathbf{R}^2 \simeq \mathbf{C}$. Now using the logarithmic potential (as one should in the plane), the electrostatic field due to point charges $1/N$ at the points ζ_j of an N-tuple Z_N in $\mathbf{C}$ has the complex representation

$$\mathcal{E}(x, Z_N) = \left\{ -\frac{1}{N} \sum_{j=1}^{N} \frac{1}{\zeta_j - z} \right\}^{-},$$

where the bar stands for complex conjugation.

Theorem 3.1 *For every N-tuple Z_N of points on $C(0,1)$,*

$$\sup_{|z|=r} |\mathcal{E}(z, Z_N)| > \frac{r^{N-1}}{2N+2}, \qquad 0 < r < 1,$$

while for the special N-tuple $\tilde{Z}_N$ consisting of the vertices of a regular N-gon inscribed in $C(0,1)$,

$$\sup_{|z|\le r} |\mathcal{E}(z, \tilde{Z}_N)| = \frac{r^{N-1}}{1-r}, \qquad 0 < r < 1. \tag{3.1}$$

Point charges $1/N$ at the vertices of a regular N-gon happen to minimize the potential energy for N-tuples on $C(0,1)$. Switching to an arbitrary Jordan curve Γ, the condition of minimal potential energy leads to an N-tuple $\tilde{Z}_N = (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N)$ of so-called Nth order Fekete points on Γ. If we limit ourselves to fields of point charges at Fekete N-tuples $\tilde{Z}_N$, circles are quite exceptional among plane curves:

Theorem 3.2 [5,2,6] *Let Γ be a Jordan curve of class $C^{3+\epsilon}$ different from a circle and let K be an arbitrary closed domain in its interior. Then for point charges $1/N$ at Nth order Fekete points $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N$ on Γ,*

$$\sup_K |\mathcal{E}(x, \tilde{Z}_N)| \sim \frac{c}{N} \qquad \text{as } N \to \infty \tag{3.2}$$

for some constant $c = c(\Gamma, K) > 0$.

Compare this result with (3.1) for the case of a circle! We expect that (3.2) has an analog for smooth surfaces with N replaced by $\sqrt{N}$, cf. [3]. Could it be that (3.1) has an analog for the sphere with N replaced by $\sqrt{N}$? Cf. Conjecture 2.3.

4 Back to the sphere

The difficulty in the case of S is that we don't know where to put our N point charges in order to get a small field. We wish to distribute them very regularly over S, but there are no regular polytopes with more than 20 vertices! Minimazation of the potential energy $V(Z_N)$ (1.1) does lead to well-distributed, well-separated N-tuples, but the corresponding fields or potentials are not readily amenable to analytic treatment. However, if we minimize certain related functions, it will become possible to apply multidimensional complex analysis. In the following we will treat one such function; another one is described in Section 6.

In order to obtain a small field $\mathcal{E}(x, Z_N)$ on balls $B(0,r)$ it is sufficient to make the potential $U(x, Z_N)$ nearly constant on such balls, or to make the adjusted potential

$$U^*(x) = U^*(x, Z_N) = U(x) - U(0) = U(x) - 1 \tag{4.1}$$

very small. Being harmonic on the unit ball, $U^*(x)$ has average zero on every sphere $S(0,r)$ with $r < 1$. Thus if we fix r and minimize the function

$$F_r(\zeta_1, \ldots, \zeta_N) = \int_S U^*(r\eta)^2 d\sigma(\eta), \qquad \zeta_j \in S \tag{4.2}$$

we may expect that $U^*(r\eta)$ will vanish at many well-distributed, well-separated points $\eta_k \in S$ In order to formulate a precise hypothesis we need a definition.

Definition 4.1 We say that the (adjusted) potential $U^*(x) = U^*(x, Z_N)$ (4.1) is of zero type (r, δ, M), where $r \in (0,1)$, $\delta > 0$ and $M \geq 10$, if the function $U^*(r\eta)$, $\eta \in S$ vanishes at M points $\eta_k \in S$, $1 \leq k \leq M$ with the following properties:

(i) every spherical cap $\Sigma \subset S$ of area $\geq (1/5)$ area S contains $\geq M/10$ points η_k;

(ii) the points η_k admit separation constant $2\delta/\sqrt{M}$:

$$|\eta_j - \eta_k| \geq 2\delta/\sqrt{M}, \qquad j,k = 1, \ldots, M, \quad j \neq k.$$

Here the numbers 5 and 10 have been chosen for convenience; they could be replaced by other constants.

Looking at the cae of the circle, the following hypothesis would seem reasonable.

Hypothesis 4.2 *For $r \in (0,1)$ there is a constant $\gamma = \gamma(r) > 0$ such that for $N \geq N_0(r)$ and N-tuples $\tilde{Z}_N = (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N)$ on S which minimize F_r (4.2), the (adjusted) potentials $U^*(x, Z_N)$ are of zero type $(r, 1, M)$ with $M \geq \gamma N$.*

5 Use of complex analysis

We have the following

Theorem 5.1 *For fixed $r \in (0,1)$ and $\delta > 0$, there are positive constants B and c such that for every (adjusted) potential $U^*(x, Z_N)$ of zero type (r, δ, M) with $M \geq 10$,*

$$\sup_{|x|=r} |U^*(x, Z_N)| \leq Be^{-c\sqrt{M}}. \tag{5.1}$$

The constants B and c depend on r and δ but not otherwise on Z_N, nor on M.

Proof. (outline). We will sketch the proof here, referring to [4] for details. Accordingly, let $U^*(x) = U^*(x, Z_N)$ be a potential of zero type (r, δ, M) with $M \geq 10$. For $U^*(r\eta)$, $\eta \in S$ we let $\eta_1, \ldots, \eta_M$ be zero points as in Definition 4.1. Setting $R = \max(r, 4/5) < 1$ we introduce the closed disc D in the (x_1, x_2)-plane given by $x_1^2 + x_2^2 \leq R^2$ and we let $\Sigma \subset S$ be the spherical cap which lies above D. Since $R \geq 4/5$, the area $(1 - \sqrt{1 - R^2})2\pi$ of Σ will be $\geq (1/5)$ area S. Hence by our hypothesis, the cap Σ and each of its rotations about the origin contain $\geq M/10$ points η_k.

Observe that for $\eta = (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) \in \Sigma$ with $(x_1, x_2, 0) \in D$,

$$\begin{aligned} U^*(r\eta) &= \frac{1}{N}\sum_{j=1}^{N}\left\{1 + r^2 - 2r(\zeta_{j1}x_1 + \zeta_{j2}x_2 + \zeta_{j3}\sqrt{1 - x_1^2 - x_2^2})\right\}^{-\frac{1}{2}} - 1 \\ &= W(x_1, x_2), \end{aligned} \tag{5.2}$$

say. For $z = (z_1, z_2)$ ranging over a suitable $\mathbf{C}^2$ neighborhood Ω of D (depending on r), we now introduce the complexified potential

$$W(z) = \frac{1}{N}\sum_{j=1}^{N}\left\{1 + r^2 - 2r(\zeta_{j1}z_1 + \zeta_{j2}z_2 + \zeta_{j3}\sqrt{1 - z_1^2 - z_2^2})\right\}^{-\frac{1}{2}} - 1. \tag{5.3}$$

We choose Ω in such a way that we can define holomorphic branches of the roots in (5.3) (taking them positive on D). We also require that $|W(z)|$ has an upper bound A on Ω independent of N. Abusing the notation, we will henceforth write $(x_1, x_2, 0) = (x_1, x_2) = x$ for points of D

By the preceding, $U^*(r\eta)$ vanishes at $s \geq M/10$ points η_k on the cap Σ; it is convenient to rename these points $\eta_1, \ldots, \eta_s$. Then by (5.2),

$$W(x) = 0 \qquad \text{for } x = \xi_k \stackrel{\text{def}}{=} (\eta_{k1}, \eta_{k2}) \in D, \qquad k = 1, \ldots, s.$$

It will follow from the hypothesis (cf. Definition 4.1) that our points $\xi_1, \ldots, \xi_s$ admit a separation constant of the form $2\delta_1/\sqrt{s}$ where $\delta_1 > 0$ depends only on δ and r (via R):

$$|\xi_j - \xi_k| \geq 2\delta_1/\sqrt{s}, \qquad j, k = 1, \ldots, s, \quad j \neq k$$

Known results of complex analysis may now be used to show that there is a constant $b > 0$ depending only on D, Ω and δ_1 (hence, only on r and δ) such that

$$|W(x)| < Ae^{-c\sqrt{M}}, \qquad \forall x \in D, \tag{5.4}$$

see the Elucidation below.

We finally return to $U^*(r\eta)$. In view of (5.2) and since $s \geq M/10$, (5.4) gives

$$|U^*(r\eta)| < Ae^{-c\sqrt{M}}, \qquad c = b/\sqrt{10}, \qquad \forall \eta \in \Sigma. \tag{5.5}$$

The same inequality will hold on every spherical cap obtained from Σ by rotation about the origin in $\mathbf{R}^3$, hence (5.5) holds for all $\eta \in S$. That is, we have (5.1).

Elucidation Inequality (5.4) may be derived from a Jensen-type theorem for $\mathbf{C}^n$ ([10] p.385) and an estimate for the area $A(\rho)$ of the zero set $Z(W)$ of W in D_ρ, the ρ-neighborhood of D in $\mathbf{C}^2$. By the Lelong-Rutishauser theorem ([10], p.386), $Z(W)$ has area $\geq \pi\delta_1^2/s$ in each of the balls $B(\xi_k, \delta/\sqrt{s})$. Since those s balls are disjoint, we find that $A(\delta_1/\sqrt{s}) \leq \pi\delta_1^2$. Now by a theorem of Berndtsson [1], $A(\rho)/\rho$ is nondecreasing, hence for $\rho \geq \delta_1/\sqrt{s}$ and as long as $D_\rho \subset \Omega$,

$$A(\rho) \geq \frac{A(\delta_1/\sqrt{s})}{\delta_1/\sqrt{s}}\rho \geq \pi\delta_1\rho\sqrt{s}.$$

This inequality is precisely what is needed for (5.4), cf. [4].

6 Conclusions

Combining Theorem 5.1 and Hypothesis 4.2 we obtain

Theorem 6.1 *Suppose that the plausible Hypothesis 4.2 is satisfied. Then for given $r \in (0,1)$, there are positive constants B and d such that for $N \geq N_0(r)$ and the special N-tuples $\tilde{Z}_N$ on S which minimize F_r (4.2),*

$$\sup_{|x|=r} |U(x, \tilde{Z}_N) - 1| \leq Be^{-d\sqrt{N}}. \tag{6.1}$$

Instead of minimizing F_r, one may minimize the simpler function

$$G_r(\zeta_1, \ldots, \zeta_N) = \frac{1}{N^2} \sum_{j,k=1}^{N} \frac{1}{|\zeta_j - r\zeta_k|}, \qquad \zeta_j \in S \tag{6.2}$$

which resembles the potential energy $V(Z_N)$ (1.1). Again denoting the minimizing N-tuples by $\tilde{Z}_N = (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N)$, the corresponding potentials $U(r\eta, \tilde{Z}_N)$ will be stationary at the points $\eta = \tilde{\zeta}_k$. Under the plausible hypothesis that the points ζ_k on S admit separation constant $1/\sqrt{N}$ one can again prove an inequality (6.1), see [4].

Given the upper bound in (6.1), the Poisson integral for $U - 1$ on the ball $B(0,r)$ may be used to obtain a similar upper bound for $\sup|\mathcal{E}(x, Z_N)|$ on slightly smaller balls. More support for Conjecture 2.3!

The Equivalence Theorem indicated in Section 2 finally shows that special N-tuples $\tilde{Z}_N$ for which one has an inequality (6.1) form good N-tuples of nodes for Chebyshev-type quadrature on S.

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On Chebyshev Polynomials over disjoint Sets

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1 Introduction

Let K be a compact subset of the real line, C be its complement $\mathbf{R} \setminus K$. Considering a Chebyshev (T-) system $\Phi = \{\phi_0, \phi_1, \ldots, \phi_N\}$ of continuous functions over K one calls a polynomial $c_0\phi_0(x) + \ldots + c_{N-1}\phi_{N-1}(x) + \phi_N(x)$ such that its uniform norm over K is the minimal one among all polynomials of such a form the Chebyshev polynomial $T_N(K, \Phi, x)$.

Those polynomials were found for the first time by P.L. Chebyshev for the cases $K = [-1; 1]$,

$$\Phi = \Phi_P = \{1, x, \ldots, x^N\} \text{ and } \Phi = \Phi_R = \{\frac{1}{\omega_N(x)}, \frac{x}{\omega_N(x)}, \cdots, \frac{x^N}{\omega_N(x)}\},$$

where $\omega_N(x) \in H_N$, the set of polynomials of degree no more than N, is fixed and non-vanishing on K. A.A. Markov gave well-known representations of $T_N(K, \Phi, x)$ for the same cases, and also for $K = [-1; 1]$,

$$\Phi = \Phi_A = \{\frac{1}{\sqrt{\omega_{2N}(x)}}, \frac{x}{\sqrt{\omega_{2N}(x)}}, \cdots, \frac{x^N}{\sqrt{\omega_{2N}(x)}}\},$$

where $\omega_{2N}(x)$ is a fixed polynomial which is positive on K, through trigonometric functions. Those representations now are known under the name "Chebyshev-Markov rational functions" (see [15] for exposition of their theory and references there). The theory of the Chebyshev polynomials for $K = [-1; 1]$ can be treated as a part of general theory of T-systems (see [8] for that theory and [7] for some aspects of the Chebyshev polynomials theory in that case).

The case of disjoint sets was firstly investigated by N.I. Ahyeser [1,2]. Particularly he gave a parametric representation of the Chebyshev polynomials for $K = [-1; a] \cup [b; 1], \Phi = \Phi_P$.

The main ideas of the solution for $K = [a_1; b_1] \cup \ldots \cup [a_p; b_p], \Phi = \Phi_P$ are analogous to ones of P.L. Chebyshev, E.I. Zolotarev, and N.I. Ahyeser. The integral representation is easy to obtain (it was suspected as a solution in [10],[13] and was written down explicitly in [14]). The parametric representation through automorphic functions which is rather similar to one of the previous case was obtained by the author (see [11] for an announcement). It needs to be mentioned that the author gave this representation for the first time in 1991 at the conference on function theory in Odessa being unaware of [14],[17]; last item also includes a nice survey of the problem with references.

In the case $\Phi = \Phi_R$ one needs to take into account the possibility of degeneracy. The non-degenerate case for $K = [-1; a] \cup [b; 1]$ was investigated in [12] (without indicating the nondegeneracy).

The main goal of this note is to consider the case $\Phi = \Phi_R, K = [a_1; b_1] \cup \ldots \cup [a_p; b_p], a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_p \leq b_p$, if $a_i = b_i, 1 \leq i \leq p$ then the relation $p \geq N + 1$ is required. A way of obtaining the exact solution in finitely many steps is described.

2 Regularity

It is more convenient from the beginning to treat the problem as one of *uniform* approximation with weight $s \in C(K), s(x) \neq 0$ for $x \in K$, of the function $f(x) = x^N$ by usual polynomials of degree no more than $N - 1$.

Proposition 1 (The Chebyshev alternation theorem) *Let* $f \in C(K), E_n = \max_{x \in K} |\frac{f(x)-p(x)}{s(x)}|$, *and* $\epsilon(x) = \frac{f(x)-p(x)}{s(x)}$. *The* $p(x)$ *is the best approximation of* f *in* $C(K)$ *by the set* H_n *iff there are at least* $n + 2$ *points* $\{x_i\}_{i=1}^{n+2} \subset K, x_i < x_{i+1}$ *where*

$$|\epsilon(x_i)| = E_n, i = 1, 2, \ldots, n + 2 \tag{1}$$

$$\epsilon(x_{i+1}) = \epsilon(x_i)(-1)^{1+\sum_{j \in J} s_j}, J = \{j : x_i \leq b_j < a_{j+1} \leq x_{i+1}\} \tag{2}$$

(Here $(-1)^{s_j} = sign\{s(b_j s(a_{j+1})\})$. The proof of this theorem is quite analogous to usual one (see, for instance, [6]) and is omitted here.

A set of points $\{x_i\}$ satisfying (1)-(2) is called an alternation set, points satisfying (1) are called deviation points. For $f(x) = x^n, s(x) = s_n(x) \in H_n, s(x) \neq 0$ with $x \in K$, $p(x)$ being the best approximation considered above, $\epsilon(x)$ will be denoted as $R_n(\mathcal{A}, K, x)$, and E_n will be denoted as $M_n(\mathcal{A}, K)$. Here $\mathcal{A}$ is a matrix of inverse values of poles:

$$\mathcal{A} = \{a_{i,n}\}_{i=1,n=p}^{i=1,n=\infty}, \; s_n(x) = \prod_{i=1}^{n}(1 - a_{i,n}x).$$

Assume for the sake of some simplifying $a_{1,n} = 0$.

There are two possibilities in the Chebyshev - Markov problem: its solution is either degenerate or non-degenerate (in usual sense of the degeneracy for rational functions). Consider at first the non-degeneracy case.

Since between any two alternation points $R_n(\mathcal{A}, K, x)$ vanishes at least one time then all zeroes of denominator of $R_n(\mathcal{A}, K, x)$ have to be real,simple, and to lie on $[a_1; b_p]$. By the same reason there must be at least one deviation point between any two of zeroes of $R_n(\mathcal{A}, K, x)$.

In non-degenerate case there are two possibilities for any pair of the zeroes which are distinguished by points of C:

1. There is no point x between them, where the inequality $|R_n(\mathcal{A}, K, x)| > M_n(\mathcal{A}, K)$ holds.
2. There exist such points. Then there are exactly two deviation points between the zeroes, and one (arbitrary) point of them belongs to a set of alternation points.

Thus there exists in non-degenerate case a system of intervals $K' \supset K$ such that $R_n(\mathcal{A}, K', x) = R_n(\mathcal{A}, K, x)$ and $|R_n(\mathcal{A}, K', x)| > |R_n(\mathcal{A}, K, y)|$ for $x \in \mathbf{R} \setminus K' = C', y \in K'$. It will be said that n-th row of the matrix $\mathcal{A}$ is regular relatively to K'. The matrix $\mathcal{A}$ will be called regular relatively to K if for every $n \geq p$ the n-th row of $\mathcal{A}$ is regular relatively to K.

Theorem 1 *The following assertions are equivalent:*

1) The matrix $\mathcal{A}$ is regular relatively to K.

2) For every $n \in \mathcal{N}, n \geq p$ there exists a partition $\{n_1, \dots, n_p\}$ of n such that

$$\begin{vmatrix} e_{i,0} & \dots & e_{i,p-2} & e_{i,p-1} \\ f_{1,0} & \dots & f_{1,p-2} & f_{1,p-1} \\ \vdots & & \vdots & \vdots \\ f_{p-1,0} & \dots & f_{p-1,p-2} & f_{p-1,p-1} \end{vmatrix} = \begin{vmatrix} e_{i,0} & \dots & e_{i,p-2} & h_i + \frac{\pi n_i}{n} \\ f_{1,0} & \dots & f_{1,p-2} & g_1 \\ \vdots & & \vdots & \vdots \\ f_{p-1,0} & \dots & f_{p-1,p-2} & g_{p-1} \end{vmatrix} \quad (3)$$
$$, i = 1, 2, \dots, p;$$

where

$$e_{i,k} = \int_{\alpha_i}^{\beta_i} \frac{x^k dx}{\sqrt{|\omega_p(x)|}}; \quad i = 1, 2, \dots, p; \quad k = 0, 1, \dots, p-1; \quad (4)$$

$$f_{i,k} = \int_{\beta_i}^{\alpha_{i+1}} \frac{x^k dx}{\sqrt{|\omega_p(x)|}}; \quad i = 1, 2, \dots, p-1; \quad k = 0, 1, \dots, p-1; \quad (5)$$

$$g_i = \sum_{j=2}^{n} \int_{\beta_i}^{\alpha_{i+1}} \frac{\sqrt{\omega_p(1/a_{j,n})}a_{j,n}}{(1-a_{j,n}x)\sqrt{|\omega_p(x)|}}dx, \quad i = 1, 2, \ldots, p-1; \tag{6}$$

$$h_i = \sum_{j=2}^{n} \int_{\alpha_i}^{\beta_i} \frac{\sqrt{\omega_p(1/a_{j,n})}a_{j,n}}{(1-a_{j,n}x)\sqrt{|\omega_p(x)|}}dx, \quad i = 1, 2, \ldots, p; \tag{7}$$

$$\omega_p(x) = \prod_{j=1}^{p}(x-\alpha_j)(x-\beta_j). \tag{8}$$

3) There exists a polynomial solution (P, Q) with real zeroes of the equation

$$P^2(x) - \omega_p(x)Q^2(x) = s_n^2(x).$$

Sketch of the proof. 1) $\Leftrightarrow$ 2).

Since $|R_n(\mathcal{A}, K, x)| > M_n(\mathcal{A}, K)$ on C and $|R_n(\mathcal{A}, K, x)| \leq M_n(\mathcal{A}, K)$ on K one can easily see that

$$\left|\frac{R_n(\mathcal{A}, K, x) + \sqrt{R_n^2(\mathcal{A}, K, x) - M_n^2(\mathcal{A}, K)}}{M_n(\mathcal{A}, K)}\right| = |F_n(x)|$$

is equal to 1 on K. Here and in (6),(7) one needs to choose the branch of square root on $\mathbf{C} \setminus K$ such that $F_n(x)$ has no zeroes in $\mathbf{C} \setminus K$, and only poles in $1/a_{1,n}, \ldots, 1/a_{n,n}$ of order 1. Further the differential equation

$$\frac{R_n'(\mathcal{A}, K, x)}{\sqrt{R_n^2(\mathcal{A}, K, x) - M_n^2(\mathcal{A}, K)}} = \frac{r_{n+p-1}(x)}{\sqrt{\omega_p(x)}s_n(x)}, \tag{9}$$

where $r_{n+p-1}(x) \in H_{n+p-1}$, is almost evident. Then after evaluating the residu in $1/a_{j,n}, j = 2, \ldots, n$, one easily obtains the relation

$$\frac{R_n'(\mathcal{A}, K, x)}{\sqrt{R_n^2(\mathcal{A}, K, x) - M_n^2(\mathcal{A}, K)}} = \frac{c}{\sqrt{\omega_p(x)}}\left(c_{p-1}(x) + \sum_{j=2}^{n} \frac{\sqrt{\omega_p(1/a_{j,n})}a_{j,n}}{(1-a_{j,n}x)}\right), \tag{10}$$

with a monic polynomial $c_{p-1} \in H_{p-1}$ and a nonzero constant c. Integrating (10) over $[a_j; b_j]$ and $[b_i; a_{i+1}]$, $i = 1, \ldots, p-1$; $j = 1, \ldots, p$ and excluding unknown coefficients of c_{p-1} from the obtained system one concludes validity of (3), where n_j equal to $\frac{1}{\pi}\Delta \arg F_n(x)$ over $[a_j; b_j], j = 1, \ldots, p$.

2) $\Leftrightarrow$ 3). Obvious from (10).

Remarks.

1. For $a_{1,n} = \cdots = a_{n,n} = 0$ system (3) is equivalent to one from [14]. In this case theorem 1 was obtained by the author independently.

2. The differential equation (10) and the corresponding integral representation for $a_{1,n} = \cdots = a_{n,n} = 0$ were suggested as a solution of the problem of finding of the Chebyshev polynomials over several intervals in [10],[13].

3. It is not possible to take a limit in (3) with $a_{j,n} \to 0$ separately in (6) and (7), though the result for $a_{1,n} = \cdots = a_{n,n} = 0$ turns out to be the same as (3) with $g_1 = \cdots = g_{p-1} = h_1 = \cdots = h_p = 0$.

4. The case $a_{1,n} = \cdots = a_{n,n} = 0$ was discussed recently in [17]. The author became aware of this work only during preparation of the manuscript.

3 Parametric representation of the Chebyshev - Markov rational functions in regular case

Firstly recall some notions from the theory of Burnside-Schottky automorphic functions (see,for instance,[3]). Let T be a complement of an even number of disjoint disks $D_1, D_2, \ldots, D_{p-1}, D'_1, D'_2, \ldots, D'_{p-1}$ in $\mathbf{C} \cup \{\infty\}$. The corresponding Schottky group is a free group generated by fractional-linear mappings $T_1, \ldots, T_{p-1}$ such that $T_i(D_i) = (\mathbf{C} \setminus D'_i) \cup \partial D'_i \cup \{\infty\}$.

H. Weber proved [18] that for any system of p intervals K of real axis there exists a conformal map of C onto some domain $T' = T \cap \{z : \Im z > 0\}$ where T is of the above-mentioned form with D_i and D'_i being symmetric relatively to the real axis, $i = 1, 2, \ldots, p-1$,and all their centres belonging to the imaginary axis. The mapping function can be written as follows:

$$x = \frac{\beta_1 - \alpha_1 \omega^2(z)}{1 - \omega^2(z)}, \omega(z) = z \prod_{j=1}^{\infty} \frac{(z - T_j(0))(\xi - T_j(\infty))}{(z - T_j(\infty))(\xi - T_j(0))}. \tag{11}$$

Here $\Gamma = \{T_0(z) \equiv z, T_1(z), \ldots\}$ is the above-mentioned Schottky group, and ξ is the image of the infinity point.

W. Burnside proved [5] that any function automorphic relatively to the group Γ can be expressed as a product of prime-functions in the form

$$\prod_{j=1}^{m} \frac{\Omega(z, y_j)}{\Omega(z, x_j)}$$

with

$$\Omega(z, y) = (z - y) \prod_i \frac{(T_i(z) - y)(T_i(y) - z)}{(T_i(z) - z)(T_i(y) - y)},$$

where now, of each pair of inverse substitutions T and T^{-1}, only one is to be taken in the infinite product, and x's and y's are the zeroes and infinities of the function in T or their homologues.

Recall also other Schottky-Burnside automorphic functions:

$$\exp \Phi_k(z) = \prod_{j=1}^{\infty} \frac{z - J_{jk}}{z - J_j}(z - Jk), J_{jk} = (T_j T_k)^{-1}(\infty), [4]$$

$$[z; \xi] = \prod_{j=0}^{\infty} \frac{z - T_j(\xi)}{z - T_j(\bar{\xi})}, [9], [16]$$

Theorem 2 *If A is regular relatively to K then the Chebyshev-Markov rational functions $R_n(\mathcal{A}, K, x)$ have the following form:*

$$R_n(\mathcal{A}, K, x) = M_n(\mathcal{A}, K) g(\prod_{j=1}^{n} [z; \xi_{j,n}] \exp\{-\sum_{k=1}^{p-1} n_k \Phi_k(z)\})$$

where $g(z) = (z + 1/z)/2$ is the Joukowski map,z and x are connected by (11), parameters ξ's are found from the system

$$\frac{1 - \omega^2(\xi_{j,n})}{\beta_1 - \alpha_1 \omega^2(\xi_{j,n})} = a_{j,n}, j = 1, 2, \ldots, n; n \geq p,$$

$\xi = \xi_{1,n}$, and the numbers n_i are from Theorem 1.

Sketch of the proof.

After the substitution (11) the function F_n from the proof of theorem 1 will be a single-valued analytic function of variable z in domain T' with boundary values belonging to the unit circle. From the Riemann-Schwarz symmetry principle it follows that this function can be extended through the circles and the real axis by the inversion. Repeating the same consideration one can extend the function up to an automorphic function relative to the corresponding Schottky group. That automorphic function has poles only in ξ's and their homologues, and zeroes in $\bar{\xi}$'s and their homologues. Thus it can be written in the form

$$\frac{\Omega(z, T_k(\xi_{1,n}))}{\Omega(z, T_l(\overline{\xi_{1,n}}))} \prod_{i=2}^{n} \frac{\Omega(z, \xi_{i,n})}{\Omega(z, \overline{\xi_{i,n}})}$$

where k, l are some natural numbers.

From the properties of the Burnside functions this expression can be transformed into

$$\prod_{i=1}^{n} [z; \xi_{i,n}] \exp\{\sum_{k=1}^{p-1} m_k \Phi_k(z)\}$$

with m_k being some integral numbers. Simple computation of the variation of argument of that expression over ∂D_i and over the real axis completes the proof.

4 General case

Let n-th row of the matrix $\mathcal{A}$ has the following form:

$$a_{1,n}, \ldots, a_{m_0,n}, a_{m_0+1,n}, \ldots, a_{m_1,n}, a_{m_1+1,n}, \ldots, a_{m_{p-1},n}$$

where $m_0, m_1, \ldots, m_{p-1}$ are numbers of poles $1/a_{i,n}$ lying outside of $[a_1; b_p]$, in $[b_1; a_2], \ldots, [b_{p-1}; a_p]$ correspondingly. Assume also $a_1 = -1, b_p = 1$.

From precedent considerations by straightforward reasonings one can find the following description of solution of the Chebyshev-Markov problem in the general case.

Step 1 For $m_0 = n$ consider the usual Chebyshev-Markov function

$$M_n(\mathcal{A}, [-1;1]) \cos \sum_{i=1}^{n} \arccos \frac{x - a_{i,n}}{1 - a_{i,n}x}.$$

If all its deviation points belong to K then we are done.

Step 2 At most one of $m_i, 1 \le i \le p-1$ does not equal to zero.

Substep 1 If $m_0 = n$ then for any $l, 1 \le l \le n-1$, and for each $b_k, k = 1, \ldots, p-1$ there exists at most one rational function $M_n(x; b_k, \beta_{l,n}, \mathcal{A})$ (see [12] for explicit formulæ through the Jacobi elliptic and theta- functions) such that n-th row of $\mathcal{A}$ is regular relatively to $[-1; b_k] \cup [\beta_{l,n}; 1]$ with corresponding partition $\{l, n-l\}$ of n.

Similarly, for any $l, 1 \le l \le n-1$, and for each $a_k, k = 2, \ldots, p$ there exists at most one rational function $M_n(x; \alpha_{l,n}, a_k, \mathcal{A})$ such that n-th row of $\mathcal{A}$ is regular relatively to $[-1; \alpha_{l,n}] \cup [a_k; 1]$ with corresponding partition $\{n-l, l\}$ of n.

If a set of alternation points of some of these functions is a part of K then we are done.

Substep 2 Only one $m_j, j \ge 1$ is not equal zero.

For any $l, 1 \le l \le n-1$ there exists at most one rational function $M_n(x; b_k, \beta_{l,n}, \mathcal{A})$ and at most one rational function $M_n(x; \alpha_{l,n}, a_k, \mathcal{A})$ such that n-th row of $\mathcal{A}$ is regular relatively to $[-1; b_k] \cup [\beta_{l,n}; 1]$ or $[-1; \alpha_{l,n}] \cup [a_k; 1]$ with partitions $\{l, n-l\}$ or $\{n-l, l\}$ of n correspondingly. If a set of alternation points of some of these functions is a part of K then we are done.

Otherwise one needs to consider matrices $\mathcal{A}'$ such that $(n-1)$-th row of $\mathcal{A}'$ coincides with n-th row of $\mathcal{A}$ without some $\frac{1}{a_{i,n}} \in [b_j; a_{j+1}]$. If the deviation points of $M_{n-1}(x; b_j, \beta_{l,n-1}, \mathcal{A}')$ with $\beta_{l,n} = a_{j+1}$, or of $M_{n-1}(x; \alpha_{l,n-1}, a_{j+1}, \mathcal{A}')$ with $\alpha_{l,n-1} = b_j$ lie on K, then the function will be the degenerate solution of the Chebyshev-Markov problem.

Step k (with usage of automorphic functions)

Substep 1 $m_0 = n$

For any partition $\{n_1, \ldots, n_{k-1}\}$ of n, and for any subset B of $k-1$ essential elements from $E = \{b_1, a_2, \ldots, a_p\}$ (i.e. for every $j, 1 \le j \le p-1$ the set $B \cap [b_j; a_{j+1}]$ contains at most one element) there exists at most one Chebyshev-Markov rational function $R_n(\mathcal{A}, K', x)$ with $K' = [-1; b_1'] \cup \ldots \cup [a_{k-1}'; 1]$, and and with the partition $\{n_1, \ldots, n_{k-1}\}$. Here $k-1$ of the ends of these intervals consist the set B, moreover $\{b_1', \ldots, b_{k-2}'\}$ can intersect E only in the points $b_1, \ldots, b_{p-1}$, and $\{a_2', \ldots, a_{k-1}'\}$ can intersect E only in the points $a_2, \ldots, a_{p-1}$. Quantities $b_1', \ldots, a_{k-1}'$ are obtained from the system (3) viewed as a system of $k-1$ nonlinear equations with $k-1$ values known from B, and other $k-1$ unknown variables from $\{\beta_1, \alpha_2, \ldots, \alpha_{k-1}\}$. The function $R_n(\mathcal{A}, K', x)$ is given then by theorem 2.

If a set of alternation points of some of these functions is a part of K then we are done.

Substep l There are exactly $l-1$ non-zero elements $m_{j_1}, \ldots, m_{j_{l-1}}$ among the $m_1, \ldots, m_{p-1}$. For any partition $\{n_1, \ldots, n_{k-1}\}$ of n, and for any subset B of $k-1$ essential elements from E such that only one point in every pair $\{a_{l_i}, b_{l_i-1}\}, l_i = m_{j_i}, i = 1, \ldots, l-1$ belongs to B, repetition of the precedent considerations gives $R_n(\mathcal{A}, K', x)$ with analogous requirements on K'. If a set of alternation points of some of these functions is a part of K then we are done.

Otherwise one needs to consider matrices $\mathcal{A}^{(1)}$ such that $(n-1)$-th row of $\mathcal{A}^{(1)}$ coincides with n-th row of $\mathcal{A}$ without some $\frac{1}{a_{i,n}} \in [b_{l_i-1}; a_{l_i}], i = 1, \ldots, l-1$. If after reiteration of this substep (or of the preceding one for the choice of a pole in $[b_{l_i-1}; a_{l_i}]$ with $m_{j_i} = 1$) the function $R_{n-1}(\mathcal{A}^{(1)}, K', x)$ with b_{l_i-1}, a_{l_i} belonging to $E' = \{b_1', a_2', \ldots, a_{k-1}'\}$ has a set of alternation points which is a part of K then $R_{n-1}(\mathcal{A}^{(1)}, K', x)$ will be the degenerate solution. Next iterations of the last consideration can be proceeded with matrices $\mathcal{A}^{(j)}$ such that $(n-j)$-th row of $\mathcal{A}^{(j)}$ coincides with $(n-j+1)$-th row of $\mathcal{A}^{(j-1)}$ without some $\frac{1}{a_{i,n}} \in [b_{l_i-1}; a_{l_i}], i = 1, \ldots, l-1$, where i equals to none of $l_{k_1}, \ldots, l_{k_{j-1}}$ which were taken for the precedent matrices $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(j-1)}$. The substep is completed with $j \le l-1$.

The step is completed with $l, l \leq \min(p-2, k)$ substeps.

Full description is completed in $k, k \leq n$ steps.

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Answer to a query concerning the mapping $w = z^{1/m}$

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Dedicated to Jaap Korevaar on his 70th birthday

For $0 < r < 1$ and $m > 0$, let $C_{r,m}$ denote the disk $|w - c_{r,m}| < \rho_{r,m}$ with center

$$c_{r,m} := \frac{(1+r)^{1/m} + (1-r)^{1/m}}{2}$$

and radius

$$\rho_{r,m} := \frac{(1+r)^{1/m} - (1-r)^{1/m}}{2}.$$

The purpose of this note is to consider a question of L. Petković who asks (paraphrasing slightly) (a) how does one establish that the image $D_{r,m}$ of $|z-1| < r$ under $w = z^{1/m}$ has diameter $2\rho_{r,m}$ and (b) how does one establish that $D_{r,m}$ lies inside $C_{r,m}$? Part (a) follows quite easily from part (b): if $D_{r,m} \subset C_{r,m}$ then $D_{r,m}$ does not have diameter greater than $2\rho_{r,m}$; on the other hand, the mapping $w = z^{1/m}$ sends the points $z = 1 \pm r$ to the points $w = (1 \pm r)^{1/m} = c_{r,m} \pm \rho_{r,m}$ so that $D_{r,m}$ can not have diameter less than $2\rho_{r,m}$. Thus we need only concern ourselves with part (b) of Petković's question. L. Petković's question originally appeared a few years ago in the (now-discontinued) Queries column of the A.M.S. Notices ([3, Query 359]) but was never answered. Petković asks the question for $m \in \mathrm{N}$, but the question makes sense for all real $m \geq 1$. We shall prove

Theorem *For $0 < r < 1$ and every real $m \geq 1$, $D_{r,m} \subset C_{r,m}$. For $m > 1$, the boundary $\partial D_{r,m}$ meets $\partial C_{r,m}$ only at the two points $(1 \pm r)^{1/m}$.*

Proof For $m = 1$, there is nothing to prove since $D_{r,m} = C_{r,m}$. So we assume $m > 1$. The boundary of $D_{r,m}$ is a simple closed curve with parametrization

$$w(\theta) = (1 + re^{i\theta})^{1/m}, \qquad 0 \leq \theta < 2\pi.$$

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This curve is tangent to the circle $\partial C_{r,m}$ in the points $c_{r,m} \pm \rho_{r,m}$. To prove that the curve $w(\theta)$ lies inside $\partial C_{r,m}$ (except for the points $c_{r,m} \pm \rho_{r,m}$), we compute its curvature.

In general, the curvature κ of a curve $w(\theta)$ in the complex plane is given by

$$\kappa = \frac{\operatorname{Im} \bar{\dot{w}}\ddot{w}}{|\dot{w}|^3}$$

where dots denote differentiation with respect to θ, see [2]. For $w(\theta) = z(\theta)^{1/m}$ with $z(\theta) = 1 + re^{i\theta}$, we compute

$$\dot{w}(\theta) = \frac{rie^{i\theta}}{m} z(\theta)^{(1/m)-1}$$

and by logarithmic differentiation

$$\ddot{w}(\theta) = \dot{w}(\theta) \left[\frac{\left(\frac{1}{m} - 1\right) rie^{i\theta}}{z(\theta)} + i \right].$$

Hence

$$\begin{aligned} \operatorname{Im} \overline{\dot{w}(\theta)}\ddot{w}(\theta) &= |\dot{w}(\theta)|^2 \operatorname{Re} \left[\frac{\left(\frac{1}{m} - 1\right) re^{i\theta}}{z(\theta)} + 1 \right] \\ &= |\dot{w}(\theta)|^2 \left[\frac{\left(\frac{1}{m} - 1\right) r(\cos\theta + r)}{|z(\theta)|^2} + 1 \right] \end{aligned}$$

and finally

$$\kappa(\theta) = \frac{\left(\frac{1}{m} - 1\right) r(\cos\theta + r) + |z(\theta)|^2}{|\dot{w}(\theta)||z(\theta)|^2} = \frac{m + r^2 + (m+1)r\cos\theta}{r|z(\theta)|^{1+1/m}}.$$

Since $m + r^2 + (m+1)r\cos\theta \geq m + r^2 - (m+1)r = (m-r)(1-r) > 0$, the curvature is strictly positive, and so the domain $D_{r,m}$ is strictly convex. A further computation gives

$$\dot{\kappa}(\theta) = \frac{-(m^2-1)(r + \cos\theta)\sin\theta}{m|z(\theta)|^{3+1/m}}.$$

We see that $\dot{\kappa}(\theta)$ has precisely four simple zeros in $[0, 2\pi)$, namely at $\theta = 0, \pi$ and $\pm \arccos(-r)$. By a result given by Blaschke [1, p. 161], see also [2, p.30], this implies that the curve $w(\theta)$ has at most four points of intersection with any circle. The circle $\partial C_{r,m}$ is tangent to $w(\theta)$ in the points $c_{r,m} \pm \rho_{r,m}$ and so there are no more points of intersection. Hence $w(\theta)$ lies either completely inside or completely outside $\partial C_{r,m}$. (Note that we have interpreted tangential intersections to count as multiple intersections in the Blaschke result. This "double counting" interpretation is indeed valid, for otherwise we could always produce a circle close to $\partial C_{r,m}$ which meets $w(\theta)$ in at least five points.)

We shall be finished if we can show that in the point $c_{r,m}+\rho_{r,m}$ the curvature of $w(\theta)$ is greater than the curvature of $\partial C_{r,m}$. So we want to show that $\kappa(0) > 1/\rho_{r,m}$, or

$$\frac{r(1+r)^{1/m}}{m+r} < \frac{(1+r)^{1/m}-(1-r)^{1/m}}{2}.$$

Rewriting this, we need to show

$$(m+r)(1-r)^{1/m} < (m-r)(1+r)^{1/m}, \qquad m>1, \quad 0<r<1. \tag{1}$$

For $r=0$, we have equality in (1). Further it is easy to check that, for $0<r<1$, the derivative with respect to r of the left hand side is less than the derivative of the right hand side. Hence the inequality (1) holds and the proof is finished. □

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HOLOMORPHIC EMBEDDINGS OF C IN $\mathbf{C}^n$

Walter Rudin

The following two theorems were proved jointly with Jean-Pierre Rosay.

TH. I. If $n \geq 3$ and $\{\lambda_k\}$, $\{p_k\}$ are discrete sequences if $\mathbf{C}$ and $\mathbf{C}^n$ respectively (without repetition) then there is a holomorphic $F : \mathbf{C} \to \mathbf{C}^n$ such that

(1) $F(\mathbf{C})$ is a closed subset of $\mathbf{C}^n$,

(2) $F'(\lambda) \neq 0$ for every $\lambda \in \mathbf{C}$,

(3) F is one-to-one, and

(4) $F(\lambda_k) = p_k$ for $k = 1, 2, 3, \cdots$.

(Properties (1), (2), (3) may be summarized by saying that F is a proper holomorphic embedding of $\mathbf{C}$ in $\mathbf{C}^n$.)

TH. II. If E is a discrete subset of $\mathbf{C}^2$ then there is a holomorphic $F : \mathbf{C} \to \mathbf{C}^2$ that satisfies (1), (2), and

$$(4') \qquad F(\mathbf{C}) \supset E.$$

Remarks.

(i) It is not known whether Th. I holds also when $n = 2$, i.e., whether the immersion proved in Th. II can be improved to an embedding.

(ii) Th. I shows that there exist proper holomorphic embeddings of $\mathbf{C}$ in $\mathbf{C}^n$ (when $n \geq 3$) which cannot be extended to holomorphic automorphisms of $\mathbf{C}^n$.

(This follows from the existence of non-tame discrete sets in $\mathbf{C}^n$. See Th. 4.5 in Trans. AMS vol. 310, 1988, p. 59.)

(iii) It is not known whether (ii) holds for $n = 2$.

HOLOMORPHIC HULLS WITH RESPECT TO INVARIANT FUNCTIONS

A.G.SERGEEV

Let D be a domain in $\mathbb{C}^n$ invariant under the action of a compact Lie group K. What is the holomorphic hull of D with respect to holomorphic functions in D invariant under K ? It's clear from the simple examples that this holomorphic hull should differ much from the usual holomorphic hull of D. Take, e.g., the ring $D = \{1 < |z| < 2\}$ in $\mathbb{C}^1$ with the action of the circle group S^1 given by rotations. Then the only S^1-invariant holomorphic functions in D are constants so they extend holomorphically across the boundary of D to all of $\mathbb{C}^1$ (note that D is a domain of holomorphy in this example).

In this article based on the papers [3],[4],[7],[10] by P.Heinzner, X.Zhou and the author we present some general assertions about holomorphic hulls with respect to K-invariant holomorphic functions and give their applications to particular K-invariant domains such as matrix Reinhardt domains and the extended matrix disc.

1. Complexifications of invariant domains of holomorphy.

Let K be a compact connected real Lie group. The **complexification** of K (cf. Hochschild [5]) is a complex Lie group $K^{\mathbb{C}}$ with a (continuous) homomorphism $i : K \to K^{\mathbb{C}}$ such that for any (continuous) homomorphism $\varphi : K \to G$ to a complex Lie group G there exists a unique holomorphic homomorphism $\psi : K^{\mathbb{C}} \to G$ such that the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\ i\ } & K^{\mathbb{C}} \\ & \varphi\searrow \quad \swarrow\psi & \\ & G & \end{array}$$

is commutative.

The complexification $K^{\mathbb{C}}$ is uniquely defined up to biholomorphic homomorphisms and the Lie algebra $\mathfrak{k}^{\mathbb{C}}$ of $K^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of K, i.e. $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$. Moreover, $K^{\mathbb{C}}$ is Stein and $i(K)$ is a totally real submanifold of $K^{\mathbb{C}}$ with $\dim_{\mathbb{R}} i(K) = \dim_{\mathbb{C}} K^{\mathbb{C}}$. If, for example, $K = S^1$ then $K^{\mathbb{C}} = \mathbb{C}^* = \mathbb{C} \setminus 0$, if $K = \mathrm{SU}(n)$ then $K^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$.

Let the group K act linearly on $\mathbb{C}^n$, i.e. the action of K on $\mathbb{C}^n$ is given by a representation $\rho : K \to \mathrm{GL}(\mathbb{C}^n)$. Then it generates a holomorphic representation $\rho^{\mathbb{C}} : K^{\mathbb{C}} \to \mathrm{GL}(\mathbb{C}^n)$, i.e. a holomorphic linear action of $K^{\mathbb{C}}$ on $\mathbb{C}^n$. Let D be a K-invariant domain in $\mathbb{C}^n$. We call its **complexification** the domain $D_{\mathbb{C}} = K^{\mathbb{C}} \cdot D$, i.e. the image of D under $K^{\mathbb{C}}$-action. This definition agrees with the general definition of the complexification of a Stein space given in Heinzner [3].

We'll show (cf. Theorems 1,2) that under some natural condition on the K-action on D its complexification $D_{\mathbb{C}}$ coincides with the holomorphic hull of D

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with respect to invariant holomorphic functions. This "natural condition" due to Heinzner [3] is defined as follows. Let D be a K-invariant domain in $\mathbb{C}^n$. It is called **orbit convex** if for any $z \in D$ and any $v \in i\mathfrak{k}$ the inclusion $\exp v \cdot z \in D$ implies $\exp(tv) \cdot z \in D$ for $0 \le t \le 1$. Here, $\exp : \mathfrak{k}^{\mathbb{C}} \to K^{\mathbb{C}}$ is the exponential mapping. This definition means, rougly speaking, that $K^{\mathbb{C}}$-orbits of points of D going in the directions of $i\mathfrak{k}$ cannot return to D after leaving it.

We have the following extension theorem for K-invariant functions.

Theorem 1. *Let D be a K-invariant orbit convex domain in $\mathbb{C}^n$. Then any K-invariant holomorphic function f on D can be extended to a $K^{\mathbb{C}}$-invariant holomorphic function $\hat{f}$ on $D_{\mathbb{C}}$. Hence, $D_{\mathbb{C}}$ is a natural holomorphic extension of D w.r. to K-invariant holomorphic functions.*

The theorem in this form was proved in Heinzner–Sergeev [4] and in a more general situation — in Heinzner [3]. In Theorem 1 one can substitute K-invariant holomorphic functions f in D by K-**equivariant** holomorphic mappings $f : D \to Y$ to some holomorphic $K^{\mathbb{C}}$-manifold Y. The assertion is still true, i.e. such a mapping extends to a $K^{\mathbb{C}}$-equivariant holomorphic mapping $\hat{f} : D_{\mathbb{C}} \to Y$. Here, a mapping $f : D \to Y$ is called K-equivariant iff $f(k \cdot z) = k \cdot f(z)$ for any $z \in D, k \in K$.

There is a partial converse to Theorem 1 proved in Heinzner [3]. Suppose that D is a K-invariant domain of holomorphy in $\mathbb{C}^n$ and Ω is a $K^{\mathbb{C}}$-invariant domain in $\mathbb{C}^n$ such that any K-equivariant holomorphic map $f : D \to V$ to a finite-dimensional representation space $V = \mathbb{C}^k$ extends to a holomorphic $K^{\mathbb{C}}$-equivariant map $\hat{f} : \Omega \to V$. Then D is orbit convex.

We know from Theorem 1 that K-invariant holomorphic functions on an orbit convex domain D extend to $D_{\mathbb{C}}$. We are interested next in the question whether $D_{\mathbb{C}}$ is the smallest domain with this property, i.e. whether $D_{\mathbb{C}}$ coincides with the holomorphic hull of D w.r. to K-invariant holomorphic functions ? The answer is positive for orbit convex domains D.

Theorem 2. *Let D be a K-invariant orbit convex domain of holomorphy in $\mathbb{C}^n$. Then $D_{\mathbb{C}}$ is also a domain of holomorphy which represents the holomorphic hull of D w.r. to K-invariant holomorphic functions.*

This theorem was proved in Heinzner-Sergeev [4] (assuming that $D_{\mathbb{C}}$ is saturated) and in Heinzner [3] in a more general setting. The proof is based on an invariant form of Cartan's theorem which reads as follows.

Cartan's theorem. *Let D be a K-invariant domain of holomorphy in $\mathbb{C}^n$ and A is a K-invariant analytic subset in D. Then any K-invariant analytic function f on A can be extended to a K-invariant holomorphic function F on D.*

For a compact Lie group K this invariant version of the Cartan's theorem follows immediately by extending f to a holomorphic function in D (using the usual Cartan's theorem) and integrating it over the group K.

2. Orbit convex domains..

Now we shall investigate more carefully the orbit-convexity condition. First we note that it can be slightly weakened without violating the assertions of Theorems 1 and 2. This "weakened" version of orbit-convexity is called orbit-connectedness (Heinzner [3]).

Denote for $z \in \mathbb{C}^n$ by $b_z : K^{\mathbb{C}} \to \mathbb{C}^n$ the orbit map $b_z(h) = h \cdot z$, $h \in K^{\mathbb{C}}$. A K-invariant domain D in $\mathbb{C}^n$ is called **orbit connected** iff the preimage $b_z^{-1}(D) = \{h \in K^{\mathbb{C}} : h \cdot z \in D\}$ is connected in $K^{\mathbb{C}}$ for any $z \in \mathbb{C}^n$.

It follows from the polar decomposition of $K^{\mathbb{C}}$ that the orbit-convexity of D implies its orbit-connectedness. The same proof as in Theorem 1 applied to K-invariant orbit connected domains in $\mathbb{C}^n$ shows that the assertion of this theorem remains true for such domains. So from the partial converse to Theorem 1 we obtain that for K-invariant domains of holomorphy the orbit-connectedness implies orbit-convexity. Hence Theorem 2 is also true for K-invariant orbit connected domains of holomorphy.

Zhou has proved recently an extension of Theorem 2 for orbit connected domains which are not holomorphically convex.

Theorem 3. *(Zhou [10]). Let D be a K-invariant orbit connected domain in $\mathbb{C}^n$. Then its holomorphic hull $E(D)$ is schlicht and orbit connected $\iff$ $E(D_{\mathbb{C}})$ is schlicht. Moreover, in this case $E(D_{\mathbb{C}}) = K^{\mathbb{C}} \cdot E(D)$.*

A class of orbit convex domains is given by so called orbit pseudoconvex domains (Heinzner–Sergeev [4]). Let D be a K-invariant domain in $\mathbb{C}^n$ given in the form $D = \{z \in \mathbb{C}^n : \varphi(z) < 0\}$ where φ is a K-invariant real C^2-smooth function on $\mathbb{C}^n$. The function φ is called orbit plurisubharmonic (w.r. to $K^{\mathbb{C}}$-action on $\mathbb{C}^n$) if the Levi form of φ is non-negative in complex directions tangent to $K^{\mathbb{C}}$-orbits in all points of D. Domains D defined by orbit plurisubharmonic functions φ are called **orbit pseudoconvex**.

Proposition. *(Heinzner–Sergeev [4]). Let D be an orbit pseudoconvex domain in $\mathbb{C}^n$. Then it is orbit convex.*

We consider next a class of orbit convex domains presented by Reinhardt domains. Recall that a Reinhardt domain D in $\mathbb{C}^n$ is the domain invariant under the action of the torus group $(S^1)^n$, i.e. satisfying to the condition $(z_1, \ldots, z_n) \in D \implies (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in D$ for all $(z_1, \ldots, z_n) \in D$ and all real $\theta_1, \ldots, \theta_n$. The complexification of D is the domain $D_{\mathbb{C}} = (\mathbb{C}^*)^n \cdot D = \{(\lambda_1 z_1, \ldots, \lambda_n z_n) : (z_1, \ldots, z_n) \in D;\ \lambda_j \in \mathbb{C}^*, 1 \le j \le n\}$ which coincides with the direct product $(\mathbb{C}^*)^r \times \mathbb{C}^{n-r}$ for some r, $0 \le r \le n$.

It's easy to show that Reinhardt domains in $(\mathbb{C}^*)^n$ are orbit convex $\iff$ they are log-convex. Such domains in $(\mathbb{C}^*)^n$ are always orbit connected. It's well known that a complete Reinhardt domain D in $\mathbb{C}^n$ is holomorphically convex if and only if it is log-convex. In particular, any complete log-convex Reinhardt domain in $\mathbb{C}^n$ is orbit convex. Here, D is complete iff with any point $(z_1^0, \ldots, z_n^0)$ it contains also the polydisc $\{(z_1, \ldots, z_n) : |z_i| \le |z_i^0|, i = 1, \ldots, n\}$. Thus the orbit-convexity of Reinhardt domains is closely related to their holomorphic convexity.

Let us consider now a generalization of Reinhardt domains to the matrix case. Denote by $\mathbb{C}^n[m \times m]$ the space of n matrix variables, i.e. a point $Z \in \mathbb{C}^n[m \times m]$ is an n-tuple $Z = (Z_1, \ldots, Z_n)$ where all Z_i, $1 \le i \le n$, are $m \times m$-matrices with complex entries. A domain $D \subset \mathbb{C}^n[m \times m]$ is called a **matrix Reinhardt domain** (Sergeev [7]) if with any point $(Z_1, \ldots, Z_n) \in D$ all points of the form $(U_1 Z_1 V_1, \ldots, U_n Z_n V_n)$ for arbitrary unitary matrices U_i, V_i, $1 \le i \le n$, also belong to D. Otherwise, matrix Reinhardt domains are the domains invariant under the natural action of the group $[\mathrm{U}(m) \times \mathrm{U}(m)]^n$ on $\mathbb{C}^n[m \times m]$.

A matrix Reinhardt domain D is, in general, not a Reinhardt domain in $\mathbb{C}^{nm^2}$ but we can always associate with D a Reinhardt open set (maybe not connected) $\operatorname{diag} D$ in $\mathbb{C}^{nm}$, namely $\operatorname{diag} D = \{(Z_1, \ldots, Z_n) \in D : Z_i$ are complex diagonal $m \times m$-matrices, $, 1 \leq i \leq n\}$.

We have the following matrix analogue of the above assertions for Reinhardt domains.

Proposition. *A matrix Reinhardt domain $D \subset [GL(m, \mathbb{C})]^n$ is orbit connected. Hence, D is orbit convex if it is a domain of holomorphy.*

This proposition is a corollary of general results on invariant domains in homogeneous spaces proved by Lasalle [6].

Passing to the case of general matrix Reinhardt domains D in $\mathbb{C}^n[m \times m]$ we have the following analogue of the above assertion for Reinhardt domains: *a complete matrix Reinhardt domain D is holomorphically convex $\Longleftrightarrow$ $\operatorname{diag} D$ is holomorphically convex $\Longleftrightarrow$ $\operatorname{diag} D$ is log-convex* (Sergeev [7]). Here, D is complete iff with any point $(Z_1^0, \ldots, Z_n^0)$ it contains also the matrix polydisc $\{(Z_1, \ldots, Z_n) : \|Z_i\| \leq \|Z_i^0\|,\ i = 1, \ldots, n\}$ where $\|Z\| = \max\{$ eigenvalues of $\sqrt{Z^* Z}\}$ is the spectral norm of a matrix Z. In fact, a stronger result is true.

Theorem 4. *Let D be a matrix Reinhardt domain in $\mathbb{C}^n[m \times m]$. Then D is holomorphically convex $\Longleftrightarrow$ $\operatorname{diag} D$ is a* **connected** *holomorphically convex Reinhardt domain in $\mathbb{C}^{mn}$.*

This theorem is proved by Bedford–Dadok [1], the sufficiency is proved independently by Zhou [9]. (Another proof of this result was proposed in Fels [2]). Bedford–Dadok [1] had also considered domains invariant under so called polar actions of classical groups and proved a similar criterion for their holomorphic convexity.

Another important example of orbit convex domains is given by the extended matrix disc.

The matrix disc is a domain Δ in the space $\mathbb{C}[2 \times 2]$ of the form $\Delta = \{Z \in \mathbb{C}[2 \times 2] : \|Z\| < 1\}$. The condition $\|Z\| < 1$ where $\|\cdot\|$ is the spectral norm of Z (cf. above) is equivalent to the positive-definiteness of the Hermitian matrix $I - Z^* Z$. The matrix disc is invariant under the action of the group $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$ given by $Z \longmapsto UZV^{-1}$, $Z \in \Delta$, $U, V \in \mathrm{SU}(2)$. The action of the complexified group $K^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}[2 \times 2]$ is given by the same formula and the complexification $\Delta_{\mathbb{C}}$ is equal to $\Delta_{\mathbb{C}} = K^{\mathbb{C}} \cdot \Delta = \{Z \in \mathbb{C}[2 \times 2] : |\det Z| < 1\}$. Note that Δ is a matrix Reinhardt domain in $\mathbb{C}[2 \times 2]$ (but not a Reinhardt domain in $\mathbb{C}^4$).

The extended matrix disc Δ_n' is defined as

$$\Delta_n' = \{(AZ_1B^{-1}, \ldots, AZ_nB^{-1}) \in \mathbb{C}^n[2 \times 2] : (Z_1, \ldots, Z_n) \in \Delta^n,\ A, B \in \mathrm{SL}(2, \mathbb{C})\}.$$

Otherwise speaking, we consider the matrix polydisc $\Delta^n = \{(Z_1, \ldots, Z_n) \in \mathbb{C}^n[2 \times 2] ; \|Z_i\| < 1,\ i = 1, \ldots, n\}$ with the diagonal action of the group $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^n[2 \times 2]$

$$(Z_1, \ldots, Z_n) \longmapsto (UZ_1V^{-1}, \ldots, UZ_nV^{-1}), \quad U, V \in \mathrm{SU}(2).$$

Then Δ^n is invariant under K and Δ'_n coincides with the image of Δ^n under the diagonal action of $K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ given by the above formula: $\Delta'_n = \Delta^n_{\mathbb{C}} = K^{\mathbb{C}} \cdot \Delta^n$. Note that Δ'_n is not a matrix Reinhardt domain (Zhou).

There was a conjecture proposed by some mathematicians and physisists asserting that the extended matrix disc is a domain of holomorphy. It is in fact a compact version of the well-known "extended future tube conjecture" from the quantum field theory (cf., e.g., Vladimirov [8]). To formulate this last conjecture we define first the extended future tube τ'_n. For that we need to substitute the matrix polydisc Δ^n in the definition of Δ'_n by the direct product of future tubes $\tau^+ \times \cdots \times \tau^+$ (n times) where $\tau^+ = \{z = x + iy \in \mathbb{C}^4 \ : \ y_1^2 > y_2^2 + y_3^2 + y_4^2, \ y_1 > 0\}$ and the $K^{\mathbb{C}}$-action on $\mathbb{C}^n[2 \times 2]$ by the diagonal action of the identity component $L_+^{\mathbb{C}}$ of the complex Lorentz group $L^{\mathbb{C}} = \mathrm{O}(4,\mathbb{C})$. In other words, the extended future tube τ'_n is defined by

$$\tau'_n = \{(\Lambda z^{(1)}, \ldots, \Lambda z^{(n)}) \in \mathbb{C}^{4n} \ : \ z^{(i)} \in \tau^+, \ i = 1, \ldots, n; \ \Lambda \in L_+^{\mathbb{C}}\}.$$

The extended future tube conjecture asserts that τ'_n *is a domain of holomorphy for any* n. It is still open for $n > 2$. However, using the above results we can prove its compact version, namely we have the following

Theorem 5. *(Heinzner-Sergeev [4]). The extended matrix disc Δ'_n is a domain of holomorphy in $\mathbb{C}^n[2 \times 2] = \mathbb{C}^{4n}$ for any n.*

The theorem follows from Theorems 1 and 2 by proving that Δ^n is a K-invariant orbit convex domain of holomorphy in $\mathbb{C}^{4n}$; the orbit convexity follows from the proposition about orbit pseudoconvex domains above.

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Wavelet Subspaces With an Oversampling Property

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Dedicated to Jaap Korevaar on the occasion of his 70th birthday.

Abstract. In the classical Shannon sampling theorem, the same sequence of functions is both orthonormal and a sampling sequence. This is not true for most wavelet subspaces in which the sampling functions and the orthonormal bases are different. However by oversampling at double the rate the property of the Shannon wavelets is extended to a much larger class which includes the Meyer wavelets. In fact together with another condition, it characterizes them.

1. Introduction.

Expansions in series of orthogonal wavelets have a number of unique properties not shared by other orthogonal expansions. Two properties that make them useful are their superior pointwise convergence [10] [6] and their localization [5]. For example, the wavelet expansion of a continuous function converges uniformly; this is not true for most other systems. If the function is zero on an interval, the coefficients corresponding to that interval will be small or even zero.

However in this work we shall concentrate on another property, the sampling property of wavelet subspaces. These are similar to the classical Shannon sampling theorem [2] which recovers a band limited function from its values on the integers

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\, S(x-n), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $S(x) = \sin \pi x/\pi x$.

In wavelet expansions a central role is played by a multiresolution approximation, i.e. a sequence $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$ each of which is a dilation of the previous one such that $V_m \subset V_{m+1}$, $m \in \mathbb{Z}$. (More details are given in section 2.) In [11] it was shown that under a weak hypothesis satisfied by most examples, each $f \in V_0$ can be represented by a sampling series similar to (1.1). However the series is not the wavelet expansion; rather $\{S(x-n)\}$ is a different Riesz basis of V_0. Also there are some important cases which don't work, e.g. splines of even order [3].

Recently Xia and Zhang [12] introduced a family of wavelets for which the father wavelet $\varphi(t)$ satisfies both the orthogonality condition $\int_{-\infty}^{\infty} \varphi(t)\varphi(t-n)dt = \delta_{0n}$ and the sampling property $\varphi(n) = \delta_{0n}$. Unfortunately, as they showed, φ cannot have compact support; and their family does not include other important families of wavelets.

In this paper we shall weaken the sampling property somewhat. Rather than look for a sampling function in V_0 we shall look for one in the dilation space V_1 and try to recover $f \in V_0$ by its values on the half integers. This gives us a sampling theorem for many additional families of wavelets composed of orthogonal sampling functions. In particular it includes the Meyer type wavelets and in fact, together with another condition, characterizes them.

This sampling property is important since it avoids integration in the approximation to $f \in L^2(\mathbb{R})$ at the finest scale

$$f_m(t) = \sum a_{mn} 2^{m/2}\varphi(2^m t^{-n}).$$

The coefficients a_{mn} can be obtained by sampling and the others at coaser scales by the decomposition algorithm [8].

2. Background in Wavelets.

The theory of orthonormal wavelet bases of $L^2(\mathbb{R})$ may be found in a number of places. Detailed introductions may be found in [3] and [5] while a more complete development in $\mathbb{R}^n$ is found in [9]. Here we present a few of the basic concepts and examples most of which we shall use later.

A wavelet basis of $L^2(\mathbb{R})$ is composed of a sequence $\{\psi_{mn}\}$ of functions given by

$$\psi_{mn}(t) = 2^{m/2}\psi(2^m t-n), \qquad m,n \in \mathbb{Z} \tag{2.1}$$

where ψ is a fixed function in $L^2(\mathbb{R})$,the "mother wavelet." Such an orthonormal basis is difficult to construct and is usually based on another function $\varphi(t)$, the "father wavelet" or scaling function. Associated with φ is a multiresolution approximation (MRA) of $L^2(\mathbb{R})$, i.e. a nested sequence of closed subspaces $\{V_m\}_{m\in Z}$ such that

$$
\begin{array}{ll}
\text{(i)} & \{\varphi(t-n)\} \text{ is an orthonormal basis of } V_0, \\
\text{(ii)} & \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R}) \\
\text{(iii)} & f \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}, \\
\text{(iv)} & \overline{\bigcup_m V_m} = L^2(\mathbb{R}).
\end{array}
\tag{2.2}
$$

In addition to the condition $\varphi \in L^2(\mathbb{R})$ we shall also require $\hat{\varphi} \in L^1(\mathbb{R})$ (at least). Here $\hat{\varphi}$ denotes the Fourier transform $\hat{\varphi}(w) = \int_{-\infty}^{\infty} \varphi(t)e^{-iwt}\,dt$. Clearly $\{2^{1/2}\ \varphi(2t-n)\}$ must be an orthonormal basis of V_1 by (iii) and (i). Since $\varphi \in V_1$, by (ii), it must have an expansion

$$\varphi(t) = \sum_k c_k \sqrt{2}\ \varphi(2t-k),\ c_k \in \ell^2. \tag{2.3}$$

This is the "dilation equation" for φ; in terms of its Fourier transform it can be expressed as

$$\hat{\varphi}(w) = \sum_k c_k\ e^{-ikw/2}\ 2^{-1/2}\ \hat{\varphi}(w/2) = m_0(w/2)\hat{\varphi}(w/2) \tag{2.4}$$

Once we have the father wavelet $\phi(t)$, we may use it to construct the "mother wavelet" $\psi(t)$. It must be chosen such that $\{\psi(t-n)\}$ is an orthonormal basis of the space W_0, given by the orthogonal complement of V_0 in V_1. Then

$$V_1 = V_0 \oplus W_0.$$

If such a $\psi(t)$ can be found, then $2^{m/2}\psi(2^{m/2}t-n) = \psi_{nm}(t)$ is an orthonormal basis of W_m, the dilation space of W_0. Indeed from (2.2) it follows that

$$\bigoplus_{m\in Z} W_m = L^2(\mathbf{R})$$

and hence $\{\psi_{n,m}\}_{n,m\in Z}$ is an orthonormal basis of $L^2(\mathbf{R})$.

The method of finding $\psi(t)$ involving the dilation equation is straightforward; $\psi(t)$ is defined as

$$\psi(t) := \sqrt{2} \sum_k c_{1-k}(-1)^k \phi(2t - k), \tag{2.5}$$

or

$$\hat{\psi}(w) = e^{-iw/2}\, \overline{m_0(\tfrac{w}{2}+\pi)}\; \hat{\varphi}(\tfrac{w}{2}). \tag{2.6}$$

Then it is merely a matter of checking that the orthogonality conditions are satisfied [5].

3. __Meyer Type Wavelets.__ This example which will appear again in the next section, is an alternate way of defining the wavelets studied originally by Lemarie and Meyer [7] and subsequently by Auscher, Weiss, and Wickerhauser [1]. They have the property that the Fourier transform of the father wavelet $\varphi(t)$ has compact support.

__**Definition 3.1.**__ Let the function $\varphi(t)$ be given by

$$\hat{\varphi}(w) = \left\{\int_{w-\pi}^{w+\pi} h\right\}^{1/2}$$

where h is a symmetric, positive distribution with support in $[-\frac{\pi}{3}, \frac{\pi}{3}]$ such that $\langle h,1\rangle = 1$. Then $\varphi(t)$ will be a father wavelet with an associated MRA $\{V_m\}$. Its associated wavelets will be denoted __Meyer type__ wavelets.

We first consider some of the properties of $|\hat{\varphi}(w)|^2$. Let $[-\varepsilon,\varepsilon]$ be the smallest interval containing the support of h. It is clear that

$$
\begin{aligned}
&\text{(i)} && \operatorname{supp}|\hat{\varphi}(w)|^2 = [-\pi-\varepsilon,\ \pi+\varepsilon] \subseteq [-\tfrac{4\pi}{3},\ \tfrac{4\pi}{3}], \\
&\text{(ii)} && |\hat{\varphi}(w)| = 1 \text{ for } |w| \le \tfrac{2\pi}{3}, \qquad\qquad (3.1)\\
&\text{(iii)} && \sum_k |\hat{\varphi}(w+2\pi k)|^2 = \int_{-\infty}^{\infty} h = 1.
\end{aligned}
$$

In order to show that φ is a father wavelet we must show the dilation equation (2.4) holds. If we define

$$m_0(\tfrac{w}{2}) := \sum_k \hat{\varphi}(w+4\pi k)$$

then $m_0(\frac{w}{2}) = 0$ for $\frac{4\pi}{3} < |w| < \frac{8\pi}{3}$ and hence

$$\hat{\varphi}(w) = m_0(\tfrac{w}{2})\hat{\varphi}(\tfrac{w}{2})$$

holds. Since by (3.1) (iii) and an application of Poisson's summation formula φ is orthogonal to its translates as well, it is a father wavelet. (The condition (2.2) (iv) is clear in this case.) The mother wavelet $\hat{\psi}$ (2.6) satisfies $|\hat{\psi}(w)|^2 = \int_{|w|/2-\pi}^{|w|-\pi} h.$

Each appropriate distribution (i.e. each probability measure with support in $[-\frac{\pi}{3}, \frac{\pi}{3}]$) will generate a MRA, which, as we shall see, has the oversampling property. We present a few particular cases for completeness.

<u>**Example 1.**</u> <u>**Shannon wavelet.**</u> Take $h(w) = \delta(w)$; then

$$\hat{\varphi}^2(w) = \int_{w-\pi}^{w+\pi} h = \begin{cases} 1, & w-\pi \le 0 \le w+\pi \\ 0, & \text{o.w.} \end{cases}.$$

This gives $\hat{\varphi}(w) = \chi_{[-\pi,\pi]}(w)$ whose inverse Fourier transform is

$$\varphi(t) = \sin \pi t/\pi t = \operatorname{sinc} t.$$

Expansions in terms of $\{\varphi(t-n)\}$ of $f \in V_0$ constitute the well known <u>Shannon sampling theorem</u> given by (1.1).

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Example 1. Shannon wavelet. Take $h(w) = \delta(w)$; then

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Expansions in terms of $\{\varphi(t-n)\}$ of $f \in V_0$ constitute the well known <u>Shannon sampling theorem</u> given by (1.1).

Example 2. Original Meyer Wavelet. Lemarie and Meyer [5] defined a wavelet whose father wavelet [5, p. 137] turns out to be

$$\hat{\varphi}(w) = \begin{cases} 1, & |w| \le \frac{2\pi}{3} \\ \cos[\frac{\pi}{2} \nu(\frac{3}{2\pi}|w|-1)], & \frac{2\pi}{3} \le |w| \le \frac{4\pi}{3} \\ 0, & \text{o.w.} \end{cases}$$

where $\nu(x)$ is a C^k function $k \ge 1$, satisfying

$$\nu(x) = \begin{cases} 0, & x \le 0 \\ 1, & 1 \le x \end{cases}$$

and $\nu(x) + \nu(1-x) = 1$.

This can be put into the form of Definition 3.1 by differention of $\hat{\varphi}(w)$.

Example 3. **Exponential function.** The standard example of a C^∞ function with compact support is

$$e(x) = \begin{cases} e^{1/(x^2-1)} & |x| < 1 \\ 0 & , 1 \le |x| \end{cases} .$$

Then $h(w) = C_\varepsilon \, e(\frac{w}{\varepsilon})$ satisfies the required conditions for $\varepsilon \le \pi/3$. The resulting $\hat{\varphi} \in C^\infty$ and $\varphi(t) \in S$.

Example 4. Non symmetric $h(w)$. Let $h(w) = \varepsilon^{-1}\chi_{[0,\varepsilon]}(w)$. The $\varphi(t)$ satisfies (2.2) but will not be real.

Example 5. **Non positive** $h(w)$. If h is taken to be

$$h(w) = \frac{3}{4}\delta(w+\varepsilon) - \frac{1}{2}\delta(w) + \frac{3}{4}\delta(w-\varepsilon)$$

then the conditions are satisfied except that h is not positive. This still however leads to a legitimate $\varphi(t)$.

4. A Characterization of Over Sampling Subspaces.

In this section we first look for the properties of the father wavelet $\varphi(t)$ in order that $\{\varphi(2t-n)\}$ be a sampling sequence for V_0. We then show that for the Meyer family, these properties are satisfied. Finally we show that the over sampling property together with another property characterize this family.

Accordingly let $\varphi(t) = O((1+|t|)^{-1-\varepsilon})$ and $\hat{\varphi}(w) = O((1+|w|)^{-1-\varepsilon})$, $\varepsilon > 0$, and let $\{\varphi(t-n)\}$ be an orthonormal basis of V_0 such that for each $f \in V_0$,

$$f(t) = \sum_n f(\tfrac{n}{2})\ \varphi(2t-n), \tag{4.1}$$

in the sense of $L^2(\mathbb{R})$. Then, in particular,

$$\varphi(t) = \sum_n \varphi(\tfrac{n}{2})\ \varphi(2t-n) \tag{4.2}$$

and by taking Fourier transforms we find

$$\begin{aligned}\hat{\varphi}(w) &= \frac{1}{2} \sum_n \varphi(\tfrac{n}{2}) e^{-iwn/2}\ \hat{\varphi}(\tfrac{w}{2}). \\ &= \sum_k \hat{\varphi}(w+4\pi k)\ \hat{\varphi}(\tfrac{w}{2}).\end{aligned} \tag{4.3}$$

The last equality is obtained by finding the Fourier coefficients of the 4π periodic function

$$\hat{\varphi}^*(w) := \sum_k \hat{\varphi}(w+4\pi k),$$

which are exactly $\frac{1}{2}\varphi(\frac{n}{2})$. Thus the Fourier transform of the dilation equation (2.4) is

$$\hat{\varphi}(w) = \hat{\varphi}^*(w)\ \hat{\varphi}(\tfrac{w}{2}). \tag{4.4}$$

This is also sufficient for (4.1) to hold.

Lemma 4.1. Let φ be a father wavelet such that $\varphi(t) = O((1+|t|)^{-1-\varepsilon})$ and $\hat{\varphi}(w) = O((1+|w|)^{-1-\varepsilon})$ for $\varepsilon > 0$; then (4.4) holds for φ if and only if (4.1) holds for all $f \in V_0$.

In order to show that (4.1) holds we must first show the series converges in the sense of $L^2(\mathbb{R})$. Since $\{\sqrt{2}\varphi(2t-n)\}$ is orthonormal, we need only show that $\{f(\frac{n}{2})\} \in \ell^2$. Since $f \in V_0 \subseteq V_1$, it has an expansion convergent in $L^2(\mathbb{R})$ and because of the decay property of $\varphi(t)$, uniformly on bounded sets,

$$f(t) = \sum_n a_{n,1}\sqrt{2}\ \varphi(2t-n).$$

Thus we have

$$f(\tfrac{k}{2}) = \sum_n a_{n,1}\sqrt{2}\ \varphi(k-n)$$

and by taking the discrete Fourier transform, we find

$$\sum_k f(\tfrac{k}{2})e^{iwk} = \sqrt{2}\sum_n a_{n,1}e^{iwn}\sum_k \varphi(k)e^{iwk}. \tag{4.5}$$

The right hand side is the product of a bounded function and an $L^2(-\pi,\pi)$ function. Hence the left hand side of (4.5) is in $L^2(-\pi,\pi)$ and $\{f(\frac{k}{2})\} \in \ell^2$.

To show that it converges to $f(t)$ we use its expansion in V_0, which, by (4.2) is

$$\begin{aligned} f(t) &= \sum_n a_{n,0}\ \varphi(t-n) \\ &= \sum_n a_{n,0}\sum_j \varphi(\tfrac{j}{2})\varphi(2t-j-2n) \\ &= \sum_n a_{n,0}\sum_\ell \varphi(\tfrac{k}{2}-n)\varphi(2t-k) \\ &= \sum_k f(\tfrac{k}{2})\varphi(2t-n). \end{aligned}$$

k

The interchange of the two series is justified since the inner series is a convolution of two ℓ^1 sequences. This is all we need since the last series converges in $L^2(\mathbb{R})$. □

We now turn to the Meyer type wavelets presented in section 3. Their father wavelets φ were given by

$$\hat{\varphi}(w) = \left\{\int_{w-\pi}^{w+\pi} h(\zeta)d\zeta\right\}^{1/2} . \tag{4.6}$$

where the smallest support interval of h is $[-\varepsilon,\varepsilon]$, $0 \le \varepsilon \le \frac{\pi}{3}$.

Lemma 4.2. Let $\hat{\varphi}(w)$ satisfy (4.6); then it also satisfies (4.4).

Since $\hat{\varphi}$ is taken to be the positive square root in (4.6), we need only show that

$$|\hat{\varphi}(w)|^2 = |\hat{\varphi}^*(w)|^2 \ |\hat{\varphi}(\tfrac{w}{2})|^2. \tag{4.7}$$

Since $\hat{\varphi}$ has support on $[-\pi-\varepsilon, \pi+\varepsilon]$, it follows that $\hat{\varphi}^*$ has support on $\Omega = \bigcup_k[-\varepsilon+(4k-1)\pi,\varepsilon+(4k+1)\pi]$. Thus on the support of $\hat{\varphi}(\frac{w}{2})$, $\hat{\varphi} = \hat{\varphi}^*$ and (4.7) becomes

$$|\hat{\varphi}(w)|^2 = |\hat{\varphi}(w)|^2|\hat{\varphi}(\tfrac{w}{2})|^2.$$

Moreover, $\hat{\varphi}(\frac{w}{2}) = 1$ on $[-2\pi+2\varepsilon,2\pi-2\varepsilon] \supseteq [-\pi-\varepsilon,\pi+\varepsilon]$, the support of $\hat{\varphi}$. Thus (4.7) holds. □

We can also go in the opposite direction. We begin with (4.4) and try to get (4.6).

Lemma 4.3. Let $\varphi(t)$ be a father wavelet satisfying the conditions of lemma 4.1 and (4.4); let the support of $\hat{\varphi}$ be a bounded interval; then there is a distribution h, $\langle h,1\rangle = 1$, with support in an interval of length $\le 2\pi/3$ contained in $[-\pi,\pi]$ such that

$$\int_{w-\pi}^{w+\pi} h \geq 0 \qquad \text{and} \qquad \hat{\varphi}(w) = \left[\int_{w-\pi}^{w+\pi}\right]^{1/2}.$$

We first observe that the support of $\hat{\varphi}$ must be a finite interval $[-a,b]$ where both a and b are positive. This follows from the fact that $\hat{\varphi}(0) = 1$ (this is true for all nice scaling functions, see [5]). Since $\hat{\varphi}$ is continuous its support contains a neighborhood of the origin.

The support of $\hat{\varphi}^*$ is $\Omega = \cup[-a+4\pi k, b+4\pi k]$ and hence if $b + a \geq 4\pi$, would be all of $\mathbb{R}$. But this is impossible since by (4.4) the support of $\hat{\varphi}(\frac{w}{2})$ would also have to be $[-a,b]$. We can say much more since by (4.4)

$$[-a,b] = [-2a,2b] \cap \Omega.$$

Hence $-a + 4\pi \geq 2b$ and $b - 4\pi \leq -2a$ which may be expressed as

$$a + 2b \leq 4\pi, \; 2a + b \leq 4\pi \tag{4.8}$$

which may be added to obtain $a + b \leq 8\pi/3$.

On the other hand $2\pi \leq a + b$ since otherwise the orthogonality condition

$$\sum_k |\hat{\varphi}(w+2\pi k)|^2 = 1, \qquad w \in \mathbb{R}. \tag{4.9}$$

would be violated.

Since on the support of $\hat{\varphi}$, (4.4) becomes

$$\hat{\varphi}(w) = \hat{\varphi}(w)\hat{\varphi}(\frac{w}{2}), \tag{4.10}$$

it follows that $\hat{\varphi}(\frac{w}{2}) = 1$ on $[-a,b]$ or $\hat{\varphi}(w) = 1$ on $[-\frac{a}{2},\frac{b}{2}]$.

We now define h(w) to be

$$h(w) := \begin{cases} -\frac{d}{dw}\,|\hat{\varphi}(w+\pi)|^2, & 0 < w + \pi \\ 0 & w + \pi \le 0 \end{cases}$$

where the derivative is in general taken in the distribution sense. It should be noticed that $h(w) = 0$ for $w < \pi - a$ or $w > b - \pi$. Furthermore, since $|\hat{\varphi}(w+\pi)|^2 + |\hat{\varphi}(w-\pi)|^2 = 1$ for $\pi - a < w < b - \pi$, it follows that

$$h(w) = \begin{cases} \frac{d}{dw}\,|\hat{\varphi}(w-\pi)|^2, & w - \pi < 0 \\ 0 & w - \pi \ge 0. \end{cases}$$

From these two expressions we deduce that

$$|\hat{\varphi}(w)|^2 = \int_{w-\pi}^{w+\pi} h(\zeta)ds \ge 0, \qquad w \in \mathbb{R}, \tag{4.11}$$

which may be rewritten as the conclusion of the Lemma, since the length of the support of h is $b + a - 2\pi \le 2\pi/3$. □

By the hypothesis $|\hat{\varphi}(w)|^2$ must be continuous but not necessarily differentiable, so that h is not necessarily a function. To be consistent with (4.10) we must take $\hat{\varphi}(w)$ to be the positive square root of $|\hat{\varphi}(w)|^2$. Its inverse Fourier transform $\varphi(t)$ is not necessarily real. However $\hat{\varphi}(w)$ is symmetric about $\frac{-a+b}{2}$ if h(w) is symmetric about 0. Therefore in this case $\varphi(t)$ can be made real by shifting w by an amount $\frac{-a+b}{2}$. Then h(w) satisfies the conditions of Definition 3.1, except for the positivity. We add this as a hypothesis to get

Theorem 4.1. Let $\varphi(t)$ be a real, symmetric father wavelet such that $\varphi(t) = O(1+|t|)^{-1-\varepsilon}$, $\hat{\varphi}(w) = O(1+|w|)^{-1-\varepsilon}$, $\varepsilon > 0$ and $\hat{\varphi}$ is non increasing for $w > 0$; then $\varphi(t)$ is a Meyer type father wavelet if and only if (i) $\varphi(t)$ satisfies the double sampling property (4.2) and (ii) the support of $\hat{\varphi}(w)$ is a bounded interval.

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Normal Families Revisited

Lawrence Zalcman

1. A matter of principle.

Let D be a domain in the complex plane $\mathbb{C}$. We shall be concerned with analytic maps (i.e., meromorphic functions)

$$f : (D, |\quad|_{\mathbb{R}^2}) \to (\hat{\mathbb{C}}, \chi)$$

from D (endowed with the Euclidean metric) to the extended complex plane $\hat{\mathbb{C}}$, endowed with the chordal metric χ, given by

$$\chi(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}} \quad z, z' \in \mathbb{C}$$
$$\chi(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}.$$

A family $\mathcal{F}$ of meromorphic functions on D is said to be *normal* on D if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence which converges χ-uniformly on compact subsets of D. It is easy to see that in case all functions in $\mathcal{F}$ are holomorphic, this condition is equivalent to the requirement that each sequence $\{f_n\} \subset \mathcal{F}$ have a subsequence which either converges uniformly (with respect to the Euclidean metric) on compacta in D or diverges uniformly to ∞ on compacta in D.

The classic example of a *non-normal* family is the collection of (holomorphic) functions $\mathcal{F} = \{nz : n = 1, 2, 3, \dots\}$ on the unit disc Δ. Indeed, set $f_n(z) = nz$ and let K be a compact subset of Δ which contains the origin and at least one other point z_1. No subsequence of $\{f_n\}$ can converge uniformly on K (since $f_n(z_1) \to \infty$), nor can any subsequence of $\{f_n\}$ diverge to infinity (as $f_n(0) = 0$ for all n). We shall have occasion to refer to this example several times in the sequel.

Normality is, quite clearly, a compactness notion: a family $\mathcal{F}$ of meromorphic functions on D is normal if and only if it is precompact in the topology of χ-uniform convergence on compact subsets of D. Accordingly, by the Arzela-Ascoli

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Theorem, normality is equivalent to equicontinuity on compacta of the functions in $\mathcal{F}$. Since these are smooth functions, this equicontinuity should be equivalent to the boundedness of an appropriate derivative. Such is the content of

Marty's Theorem. [18] *A family $\mathcal{F}$ of functions meromorphic on D is normal on D if and only if for each compact subset $K \subset D$ there exists a constant $M(K)$ such that*

$$f^{\#}(z) \leq M(K) \tag{*}$$

for all $z \in K$ and all $f \in \mathcal{F}$.

Here $f^{\#}$ denotes the *spherical derivative*

$$\begin{aligned} f^{\#}(z) &= \lim_{h \to 0} \frac{\chi(f(z+h), f(z))}{|h|} \\ &= \frac{|f'(z)|}{1+|f(z)|^2} \qquad\qquad (f(z) \neq \infty). \end{aligned}$$

Since $\chi(z,w) = \chi(1/z, 1/w)$, $f^{\#} = (1/f)^{\#}$, which provides a convenient formula for $f^{\#}$ at poles of f.

Marty's Theorem provides a complete and satisfying answer to the question of when a family of functions is normal. Unfortunately, in practice it is almost useless, as verification of the condition (*) in cases when normality is not evident is generally extremely difficult. The search for other, more useful, conditions for normality has given rise to the following *heuristic principle*: "A family of holomorphic (meromorphic) functions which have a property P in common in a domain D is (apt to be) a normal family in D if P cannot be possessed by nonconstant entire (meromorphic) functions in the finite plane" [10, p.250]. This principle is often attributed to André Bloch; however, I have been unable to find any mention of it in his published writings.

Examples.

1. Say f has property P on D if $|f(z)| \leq 17$ for $z \in D$. By Liouville's Theorem, any entire function with P is constant. That a family of (holomorphic) functions having P on a domain D is normal is a familiar result of Montel.

2. Say f has P on D if $f(z) \neq a, b, c$ on D, where a, b, c are (distinct) fixed values in $\hat{\mathbb{C}}$. Picard's Little Theorem asserts that a meromorphic function on $\mathbb{C}$ with this property must be constant, and it is a celebrated theorem of Montel that a family of meromorphic functions having P on a domain D is normal.

3. Say f has P on D if f is univalent on D, and $f(z) \neq a, b$ on D, where a and b are (distinct) fixed values in $\hat{\mathbb{C}}$. We leave the details to the reader.

The heuristic principle has proved itself extremely effective in the identification of criteria which insure normality. However, it must be used with care. Consider, for example, the property "f is not entire." Obviously, no nonconstant entire function has this property (nor, for that matter, do the constants). On the other hand, it is clear that the property does not imply normality. Indeed, fix an analytic function g having the unit circle as a natural boundary and consider the collection of functions $\mathcal{F} = \{g_n\}$ on Δ, where $g_n(z) = nz + g(z)$. Clearly, no function in $\mathcal{F}$ is entire; equally clearly, $\mathcal{F}$ is not a normal family.

If the (counter)example of the previous paragraph seems frivolous (and I do not think it is), more serious examples are near at hand. For example, the property "f is bounded" forces an entire function to be constant; however, every function in our paradigm non-normal family $\{nz\}$ has this property on Δ. A similar comment applies to the property "f omits three values." Examples like these point up the need for a *rigorous* version of the heuristic principle. In his retiring presidential address to the Association for Symbolic Logic [27], Abraham Robinson listed this as one of twelve problems worthy of attention of logicians (and, by extension, of mathematicians in general).

It turns out to be possible to answer Robinson's question in a stronger form than he had expected and without recourse to nonstandard analysis (which, naturally enough, he had hoped might provide the key to the solution). However, before turning to this, we need to dispose of the question: what is a property? Answer: a set (*viz.*, the set of all objects having the property!). As we shall be concerned with properties of functions on domains in the plane, it will be convenient to display the domain of the function explicitly together with the function. Thus, following Robinson, we write $\langle f, D\rangle$ to denote the function f defined on the domain $D \subset \mathbb{C}$, and we distinguish between the functions $\langle f, D\rangle$ and $\langle f, D'\rangle$ if $D \neq D'$. If the function f has property P on D, we write $\langle f, D\rangle \in P$. It is now possible to state

Theorem 1. [36] *Let P be a property of meromorphic (holomorphic) functions which satisfies the following three conditions:*

(i) If $\langle f, D\rangle \in P$ and $D' \subset D$, then $\langle f, D'\rangle \in P$.

(ii) If $\langle f, D\rangle \in P$ and $\varphi(z) = az + b$, then $\langle f \circ \varphi, \varphi^{-1}(D)\rangle \in P$.

(iii) Let $\langle f_n, D_n\rangle \in P$, where $D_1 \subset D_2 \subset D_3 \subset \ldots$ and $\bigcup_{n=1}^{\infty} D_n = \mathbb{C}$. If $f_n \to f$ χ-uniformly on compact subsets of $\mathbb{C}$, then $\langle f, \mathbb{C}\rangle \in P$.

Suppose (a) *$\langle f, \mathbb{C}\rangle \in P$ only if f is constant.*

Then (b) *$\{f : \langle f, D\rangle \in P\}$ is normal on D for each $D \subset \mathbb{C}$.*

Conversely, if (i) and (ii) hold, then (b) implies (a).

This result provides a highly satisfactory explication of the heuristic principle so far as properties formulated in terms of the values taken on or omitted by functions is concerned. In such cases, conditions *(i)* and *(ii)* will generally be satisfied trivially, while *(iii)* follows more or less routinely from Hurwitz's Theorem.

Examples. (continued)

4. Fix ε (small) and let $\langle f, D\rangle \in P$ if $f(z) \neq a, b, c$ on D, where now $a, b, c \in \hat{\mathbb{C}}$ are allowed to vary with f but $\chi(a,b)\chi(b,c)\chi(c,a) \geq \varepsilon$. By Picard's Little Theorem, *(a)* holds; hence we obtain a sharpening of the classical version of Montel's Theorem (cf. [3, p.202]).

5. Fix $a, b, c \in \hat{\mathbb{C}}$ (distinct) and natural numbers ℓ, m, n such that $1/\ell + 1/m + 1/n < 1$. Let $\langle f, D\rangle \in P$ if all a-points of f in D have multiplicity at least ℓ, all b-points multiplicity at least m, and all c-points multiplicity at least n. It is an easy consequence of Nevanlinna's Second Main Theorem [23, pp.277-281]

that if $\langle f, \mathbb{C}\rangle \in P$, then f must be constant. Thus *(a)* holds, and we obtain a generalization of a result of Montel [21, pp.125-126] due to Drasin [6, pp.238-239].

6. Let a_i $(\leq i \leq 5)$ be five distinct values in $\hat{\mathbb{C}}$. Say $\langle f, D\rangle \in P$ if all a_i-points of f in D have multiplicity at least 2, $1 \leq i \leq 5$. As in the previous example, a globally defined meromorphic function with this property must be constant. Accordingly, the family of all meromorphic functions having P on a fixed domain D is normal on D. (For holomorphic functions, three values suffice.)

7. Say $\langle f, D\rangle \in P$ if $f = g'$, where g is (analytic and) univalent on D or $f \equiv 0$ on D. (This last possibility is required if *(iii)* is to hold, as the limit of univalent functions could be constant.) Since the only univalent entire functions are linear, it is clear that $\langle f, \mathbb{C}\rangle \in P$ implies f is constant. Thus the (non-normalized) family of derivatives of all univalent functions on $D \subset \mathbb{C}$ is normal on D. By way of contrast, the family of all univalent functions on a domain is *not* a normal family (consider $\{nz\}$ on Δ), nor is the collection of *second* derivatives of univalent functions (cf. Example 10 below).

One attractive aspect of Theorem 1 is that it explains the failure of the heuristic principle in those cases when it does not give correct results.

Examples. (continued)

8. Say $\langle f, D\rangle \in P$ if f is bounded on D. Clearly *(i)*, *(ii)* and *(a)* hold in this case; however, as we have seen, *(b)* does not follow. This is because *(iii)* does not obtain. Indeed, fix *any* nonconstant entire function f and let $D_n = \{|z| < n\}$. Then $\langle f, D_n\rangle \in P$ for each n but clearly $\langle f, \mathbb{C}\rangle \notin P$. A similar discussion applies to the properties "f omits 3 (distinct) values on D" and "f is not entire" (i.e., $\langle f, D\rangle \in P \Leftrightarrow D \neq \mathbb{C}$).

9. Say $\langle f, D\rangle \in P$ if f is analytic on D and satisfies $|f(z)| \leq |f'(z)|$ and $0 \in f(D)$. Suppose $\langle f, \hat{\mathbb{C}}\rangle \in P$. Then $|f(z)| \leq |f'(z)|$ on $\mathbb{C}$, so f/f' is constant, and $(\log f)' = f'/f$ is also. Hence $f(z) = Ke^{az}$ and, since $0 \in f(D)$, $K \equiv 0$. Thus the only entire function having P is $f(z) \equiv 0$. That P does not force normality is

evident from the family $\{nz\}$ on Δ. In this example P fails to satisfy *any* of the conditions *(i)*, *(ii)*, and *(iii)*.

10. Define $\langle f, D\rangle \in P$ if $f = g''$, where g is analytic and univalent on D. The only entire function with this property is $f(z) \equiv 0$. Setting $g_n(z) = n(z + z^2/100 + z^3/100)$, we have $\operatorname{Re} g_n'(z) > 0$ on Δ so g_n is univalent there. Since $f_n(z) = g_n''(z) = n(2/100 + 6z/100)$ vanishes at $z = -1/3$ for each n, $\{f_n\}$ does not form a normal family on Δ. Clearly *(i)* and *(ii)* hold, so it must be *(iii)* that fails. Verifying this is an amusing exercise, which we leave to the interested reader.

11. Say $\langle f, D\rangle \in P$ if f is analytic on D and $f'(z) \neq -1$, $f'(z) \neq -2$, $f'(z) \neq f(z)$ for $z \in D$ or $f \equiv 0$ on D. Suppose $\langle f, \mathbb{C}\rangle \in P$; then f' is entire, hence constant (since $f' \neq -1, -2$ on $\mathbb{C}$). Thus $f(z) = az + b$. But then $f(z) - f'(z) = az + (b - a) \neq 0$, so that $a = 0$ and f is constant. However, $\{nz\}$ has P on Δ. Here it is condition *(ii)* that fails. It is obvious that *(i)* holds. To verify *(iii)*, suppose $f_n \to f$ uniformly on compacta, where $f_n' \neq -1, -2$ and $f_n' - f_n \neq 0$ on D_n. Then, by Hurwitz's Theorem, f must satisfy

	A		*B*	
1.	$f' \neq -1$	or	$f' \equiv -1$	(i.e., $f(z) = -z + b$)
2.	$f' \neq -2$	or	$f' \equiv -2$	(i.e., $f(z) = -2z + c$)
3.	$f' - f \neq 0$	or	$f' - f \equiv 0$	(i.e., $f(z) = Ke^z$)

$1B$ and $2B$ contradict 3, and $3B$ contradicts 1 and 2 unless $K = 0$. Thus $\langle f, \mathbb{C}\rangle \in P$.

These last two examples of the failure of the heuristic principle are due to Rubel [29]. They show clearly that neither *(ii)* nor *(iii)* may be dispensed with in the formulation of Theorem 1.

Theorem 1 is a fairly direct consequence of the following Lemma, which provides a characterization of *non*-normal families on Δ.

Lemma. [36] *A family $\mathcal{F}$ of functions meromorphic (resp. holomorphic) on the unit disc Δ is not normal if and only if there exist*

(a) a number $0 < r < 1$

(b) points z_n, $|z_n| < r$

(c) functions $f_n \in \mathcal{F}$

(d) numbers $\rho_n \to 0+$

such that

(e) $f_n(z_n + \rho_n \zeta) \to g(\zeta)$

χ-uniformly (resp. uniformly) on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic (resp. entire) function on $\mathbb{C}$, which can be taken to satisfy

$$g^{\#}(z) \leq g^{\#}(0) = 1, \quad z \in \mathbb{C}.$$

Examples. (continued)

12. Consider the non-normal family $\mathcal{F} = \{(2z)^n\}$ on Δ. Choose $f_n(z) = (2z)^n$, $z_n = 1/2$, $\rho_n = a/2n$. Then $f_n(z_n + \rho_n\zeta) = (1 + a\zeta/n)^n \to e^{a\zeta}$ on $\mathbb{C}$. For the normalization $g^{\#}(z) \leq g^{\#}(0) = 1$, choose $a = 2$.

13. Let $\mathcal{F} = \{nz^2\}$. Choose $f_n(z) = nz^2$, $z_n = b/\sqrt{n} + o(1/\sqrt{n})$, $\rho_n = a/\sqrt{n} + o(1/\sqrt{n})$. Then $f_n(z_n + \rho_n\zeta) = n\left(\frac{a\zeta + b + o(1)}{\sqrt{n}}\right)^2 \to (a\zeta + b)^2$ on $\mathbb{C}$. To normalize, set $a = \frac{2}{3}\sqrt[4]{3}$, $b = \sqrt[4]{1/3}$.

To prove the Lemma, suppose first that $\mathcal{F}$ is normal on Δ and that *(a)–(e)* hold. By Marty's Theorem, there exists $M > 0$ such that

$$\max_{|z| \leq (1+r)/2} f^{\#}(z) \leq M$$

for all $f \in \mathcal{F}$. Fix $\zeta \in \mathbb{C}$. For large n, $|z_n + \rho_n\zeta| \leq (1+r)/2$ so that $\rho_n f_n^{\#}(z_n + \rho_n\zeta) \leq \rho_n M$. Thus, for all $\zeta \in \mathbb{C}$, $g^{\#}(\zeta) = \lim \rho_n f_n^{\#}(z_n + \rho_n\zeta) = 0$. It follows that g is a constant (possibly infinity).

Conversely, if $\mathcal{F}$ is *not* normal, by Marty's Theorem there exists a number r^*, $0 < r^* < 1$, points z_n^* in $\{|z| \leq r^*\}$, and functions $f_n \in \mathcal{F}$ such that $f_n^{\#}(z_n^*) \to \infty$.

Fix r, $r^* < r < 1$, and let

$$M_n = \max_{|z|\le r}\left(1 - \frac{|z|^2}{r^2}\right) f_n^{\#}(z) = \left(1 - \frac{|z_n|^2}{r^2}\right) f_n^{\#}(z_n).$$

The maximum exists since $f_n^{\#}$ is continuous for $|z| \le r$. Clearly we have $M_n \ge (1 - |z_n^*|^2/r^2) f_n^{\#}(z_n^*) \to \infty$. Put

$$\rho_n = \frac{1}{M_n}\left(1 - \frac{|z_n|^2}{r^2}\right) = \frac{1}{f_n^{\#}(z_n)}.$$

This works! (For details see [36].)

The proof of the Lemma is one of the few really effective uses of Marty's Theorem of which I am aware. It is of such a general character that it is rather easily adapted to other situations. Appropriate versions thus hold for families of quasiregular or quasimeromorphic functions in space [19] and certain holomorphic mappings in $\mathbb{C}^n$ [1], leading to versions of Theorem 1 for these classes. One almost immediate consequence of the (proof of the) Lemma is

Brody's Theorem. [2] *A compact complex manifold is hyperbolic if and only if it contains no complex lines.*

For details on this, see [31, p.95].

2. Divide and conquer.

Theorem 1 works wonderfully well for properties involving values taken on or omitted by the functions in a family, but it is much less successful in dealing with properties involving derivatives. This is because such properties will not generally satisfy the requirement *(ii)* of invariance with respect to linear change of the independent variable. On the other hand, the heuristic principle is known to give correct resuts for many such properties. What is going on?

Example. 14. Say $\langle f, D\rangle \in P$ if f is analytic on D and $f \ne 0$, $f' \ne 1$ on D or $f \equiv 0$ on D. It is an old result (due, perhaps, to Borel) that an entire function with this property must be constant. It has also been known for over half a century [20] that P implies normality. However, Theorem 1 does not apply since *(ii)* fails.

In his problem book [8], Hayman listed a number of open questions on normal families of similar shape; in each case, a property involving values of certain derivatives of a function is known to imply that an entire function or a globally defined meromorphic function must be constant. Does the same property imply normality for a family of holomorphic or meromorphic functions? Over the years, most of these questions have been answered (affirmatively), frequently by a judicious mixture of Nevanlinna theory with the Lemma of the previous section. What has been missing until now has been a uniform approach which yields transparent proofs of such results with a minimum of effort, along the lines of the examples of the previous section. Now such an approach is available, thanks to the efforts of a number of Chinese mathematicians, most notably Xue-cheng Pang [25],[26].

We have already seen (in Example 11) that condition *(ii)* cannot be dispensed with in Theorem 1. Pang finds a ***substitute*** condition *(ii')* (actually a continuum of substitutes), better adapted for handling derivatives, under which (the analogue of) Theorem 1 remains true. Specifically, condition *(ii)* is replaced by

(ii') For some α, $-1 < \alpha < 1$, if $\langle f, D\rangle \in P$ and $\varphi(z) = az + b$ then
$$\langle \frac{f \circ \varphi}{a^{\alpha}}, \varphi^{-1}(D)\rangle \in P.$$

Of course, when $\alpha = 0$ this reduces to *(ii)*.

The proof of Pang's result is not difficult. It is based on an analogue of the Lemma, in which *(e)* is replaced by

(e') $\dfrac{f_n(z_n + \rho_n\zeta)}{\rho_n^{\alpha}} \to g(\zeta)$.

Here, as before, $-1 < \alpha < 1$. It turns out [4] that if all zeros of functions in $\mathcal{F}$ have multiplicity greater than or equal to k one may take $-1 < \alpha < k$; in particular, if functions in $\mathcal{F}$ ***never*** vanish, one can choose any $\alpha > -1$ [32]. Naturally, these wider possibilities of choosing α carry over to condition *(ii')*.

Exercise. Based on these last remarks show that for ***analytic*** (i.e., holomorphic) functions one can choose any $\alpha < 1$. Give an example to show that $\alpha \geq 1$ is not allowed.

Examples. (continued)

15. (Problems 5.12 and 5.13 of [8]) It is known that the condition $f'f^n \neq 1$ forces an entire function to be constant (Hayman [7], Clunie [5] for $n = 1$). The same condition implies that a meromorphic function on $\mathbb{C}$ is constant when $n \geq 2$ (Hayman [7], Mues [22] for $n = 2$). The corresponding normality results are due to Yang and Chang [33] (for analytic functions, $n \geq 2$) and [34] (for meromorphic functions, $n \geq 5$), Gu [12] (for meromorphic functions, $n = 3, 4$), and Oshkin [24] (for analytic functions n=1); cf. Li and Xie [16]. Pang's extension of Theorem 1 yields an astonishingly simple proof of all these results ***plus*** the (previously unknown) case of meromorphic functions $n = 2$.

Accordingly, let us say $\langle f, D\rangle \in P$ if $f'f^n \neq 1$ on D. It suffices to show that, for an appropriate choice of α, P satisfies *(i)*, *(ii')* and *(iii)*. Clearly, *(i)* holds. Let $g(z) = f(az+b)/a^{1/(n+1)}$. Then

$$\begin{aligned} g'(z)g^n(z) &= \frac{af'(az+b)}{a^{1/(n+1)}} \cdot \frac{f^n(az+b)}{a^{n/(n+1)}} \\ &= f'(az+b)f^n(az+b) \neq 1, \end{aligned}$$

so *(ii')* is satisfied with $\alpha = 1/(n+1)$. To verify *(iii)*, suppose $f_k \to f$ locally (χ-)uniformly, where $f_k' f_k^n \neq 1$. If $f'f^n \neq 1$, we're done. Otherwise, $f'f^n \equiv 1$ by Hurwitz's Theorem; so $f^{n+1}(z) = (n+1)z + c$, an impossibility (since f is single-valued).

Note that the above proof does not distinguish between analytic and meromorphic functions. In fact, the only thing preventing it from being a proof of normality in the (still unsettled) case $n = 1$ for meromorphic functions is the lack of a corresponding global result: it remains an open question whether a meromorphic function on $\mathbb{C}$ for which $f'f \neq 1$ must be constant.

16. The condition $(f^n)^{(k)} \neq 1$ forces an entire function to be constant when $n \geq k+1$ and a meromorphic function on $\mathbb{C}$ to be constant when $n \geq k+3$ (Hennekemper [9]). Schwick [30] has shown that these conditions imply normality.

This follows as in the previous example. Indeed, writing $\langle f, D\rangle \in P$ if $(f^n)^{(k)} \neq 1$ on D, we see at once that *(i)* is satisfied. Setting $g(z) = f(az+b)/a^{k/n}$, we have

$$(g^n)^{(k)}(z) = \left[\frac{f^n(az+b)}{a^k}\right]^{(k)} = (f^n)^{(k)}(az+b) \neq 1,$$

so *(ii′)* holds for $\alpha = k/n$ (< 1). To prove *(iii)*, suppose $f_j \to f$ locally (χ-)uniformly, and $(f_j^n)^{(k)} \neq 1$. If $(f^n)^{(k)} \neq 1$, we are done. Otherwise, $(f^n)^{(k)} \equiv 1$ by Hurwitz's Theorem, so $f(z) = \sqrt[n]{P(z)}$, where P is a nonconstant polynomial of degree $k < n$, an impossibility.

With a bit of additional work, it can be shown that one has normality for families of meromorphic functions even when $n = k+2$ [11].

17. (Problem 5.11 of [8]) A meromorphic function on $\mathbb{C}$ which satisfies $f \neq 0$, $f^{(\ell)} \neq 1$ must be a constant (Hayman [7]). A family of meromorphic functions with this property is normal (Gu [13]); for analytic functions, this goes back to Miranda [20]. For a proof, define $\langle f, D\rangle \in P$ if $f \neq 0$, $f^{(\ell)} \neq 1$ on D or $f \equiv 0$ on D. Clearly, *(i)* obtains. Putting $g(z) = f(az+b)/a^{\ell}$, we have

$$g(z) \neq 0 \qquad g^{(\ell)}(z) = f^{(\ell)}(az+b) \neq 1,$$

so that *(ii′)* holds with $\alpha = \ell$. (Note that we may choose α as large as we like since $f \neq 0$.) Finally, suppose $f_k \to f$ locally χ-uniformly. If $f \neq 0$, $f^{(\ell)} \neq 1$ or $f \equiv 0$, we are done. The only remaining possibility is that $f \neq 0$ but $f^{(\ell)} \equiv 1$, in which case f is a nonconstant polynomial and we obtain a contradiction to the fundamental theorem of algebra.

18. (Problem 5.14 of [8]) Let $a, b \in \mathbb{C}$, $a \neq 0$. Hayman [7] proved that a meromorphic function on $\mathbb{C}$ which satisfies $f' - af^n \neq b$ must be constant if $n \geq 5$; if f is entire, it suffices to take $n \geq 3$. On the other hand, Mues [22] gave examples of nontrivial meromorphic functions which satisfy $f' - af^n \neq b$ for $n = 3, 4$. For analytic functions, the normality result corresponding to Hayman's theorem was proved by Drasin [6]. The corresponding result for meromorphic functions was

established (independently) by Langley [14], Song-ying Li [17], and Xianjin Li [17]; cf. Li and Xie [16].

In this case, Pang's generalization of Theorem 1 does not seem to apply directly. However, his modified version of the Lemma yields normality *even when* $n = 4$! Indeed, let $a, b \in \mathbb{C}$, $a \neq 0$, and $n \geq 4$ be fixed. If the family of meromorphic functions which satisfy $f' - af^n \neq b$ on Δ is *not* normal, taking $\alpha = -1/(n-1)$ in *(e')*, we can find sequences $\{z_n\}$, $\{f_n\}$, and $\{\rho_n\}$ as in *(b)*, *(c)*, and *(d)* of the Lemma such that $g_j(\zeta) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j\zeta)$ converges locally χ-uniformly to a nonconstant meromorphic function g. Suppose that $g' - ag^n$ never vanishes. Putting $G = 1/g$, we obtain $G'G^{n-2} \neq -a$. Since $n - 2 \geq 2$, this implies that G is constant by results of Hayman and Mues (cf. Example 15). Thus g is constant, a contradiction. It follows that $g' - ag^n$ must vanish somewhere in $\mathbb{C}$. On the other hand

$$\begin{aligned} g_j'(\zeta) - ag_j^n(\zeta) &= \rho_j^{\frac{n}{n-1}} f_j'(z_j + \rho_j\zeta) - a\rho_j^{\frac{n}{n-1}} f_j^n(z_j + \rho_j\zeta) \\ &= \rho_j^{\frac{n}{n-1}} \{f_j'(z_j + \rho_j\zeta) - af_j^n(z_j + \rho_j\zeta)\} \neq \rho_j^{\frac{n}{n-1}} b, \end{aligned}$$

so that $g_j'(\zeta) - ag_j^n(\zeta) - \rho_j^{\frac{n}{n-1}} b \neq 0$. Since $g_j' - ag_j^n \to g' - ag^n$ uniformly on compacta disjoint from the poles of g and $\rho_j^{\frac{n}{n-1}} b \to 0$, we obtain via Hurwitz's Theorem $g' - ag^n \equiv 0$. It follows (since $g \not\equiv 0$) that

$$\frac{1}{1-n} \, \frac{1}{g^{n-1}(\zeta)} = a\zeta + c$$

so

$$g(\zeta) = \frac{1}{\sqrt[n-1]{(1-n)(a\zeta + c)}} \, ,$$

an impossibility.

Note that we have used $n \geq 4$ only in dispensing with the possibility that $g' - ag^n$ never vanishes; the second part of the proof requires only $n \geq 3$. For families of *analytic* functions, a very similar argument, using the result of Hayman's cited in Example 17, shows that the condition $f' - af^2 \neq b$ implies normality; cf. [35]. It remains an open question whether the condition $f' - af^3 \neq b$ implies normality for families of meromorphic functions.

Perhaps a word of explanation is in order concerning why the previous result for $n = 4$ holds even in the presence of nontrivial meromorphic functions satisfying $f' - af^4 \neq b$ and why it does not contradict the existence of such functions. The main point is that, in applying the Lemma, we pass from the condition $f' - af^4 \neq b$ to $g' - ag^4 \equiv 0$; and this last condition *does* imply (as we have seen) that g must vanish identically. In point of fact, Mues' construction of functions satisfying $f' - af^4 \neq b$ requires that $b \neq 0$. No contradiction to (the converse part of the extended version of) Theorem 1 is obtained, since if $b \neq 0$ the property defined by the condition $f' - af^4 \neq b$ does not satisfy *(ii')* for any value of $\alpha \in (-1,1)$ (or, for that matter, any other value of α).

The approach illustrated in this section has a number of advantages. It yields results otherwise attainable only with great effort with relative ease. Equally important, it makes no distinction between the case of analytic functions and the (formerly much more difficult) case of meromorphic functions. Moreover, it puts the work in proving normality criteria where (we think) it really belongs: in establishing results on globally defined individual functions. Last but not least, it gives a satisfying explanation of why the heuristic principle works — when it does.

3. Picard's Theorem with a smile.

Finally, as an indication of the versatility of the Lemma, let us show how it can be used to give a new elementary proof of the Great Picard Theorem. Actually, we shall prove Montel's theorem, that a family of meromorphic functions all of which omit the same three values on a domain is normal there. The well-known deduction of Picard's Theorem from this result takes only a couple of lines.

Montel's Theorem. *The collection $\mathcal{F}$ of all meromorphic functions which omit three fixed values $a, b, c \in \hat{\mathbb{C}}$ on a domain $D \subset \mathbb{C}$ is a normal family on D.*

Proof. (Cf. [28]) Since normality is a local notion, we may suppose that $D = \Delta$, the unit disc. Composing with a linear fractional transformation, we may also assume that the omitted values are $0, 1, \infty$. Let us denote by $\mathcal{F}_n$ the collection of functions on Δ which omit the values 0, ∞ and all nth roots of 1, so that $\mathcal{F} = \mathcal{F}_1$. Note that $f \in \mathcal{F}$ implies $\sqrt[n]{f} \in \mathcal{F}_n$, while if $h \in \mathcal{F}_n$ then $h^n \in \mathcal{F}$.

Suppose now that $\mathcal{F}$ is *not* normal. Then none of the families $\mathcal{F}_n$ is normal, so by the Lemma we have, for each n, a nonconstant entire function g_n obtained as a limit of functions omitting all values in $S_n = \{0, 1, e^{2\pi ik/n},\ k = 0, 1, \ldots, n-1\}$. By Hurwitz's Theorem g_n also omits S_n. Moreover, $g_n^{\#}(z) \leq g_n^{\#}(0) = 1$.

Write, for convenience, $T_n = S_{2^n}$, $G_n = g_{2^n}$, and consider the family $\mathcal{G} = \{G_n\}$ on $\mathbb{C}$. Now $G_n^{\#}(z) \leq 1$ for all $z \in \mathbb{C}$, so by Marty's Theorem $\mathcal{G}$ is normal on $\mathbb{C}$; hence a subsequence converges, χ-uniformly on compacta, to a limit function G. Since $G_n^{\#}(0) = 1$ for all n, $G^{\#}(0) = 1$, so G is nonconstant. The sets T_n are nested, so that G_m omits values in T_n as soon as $m \geq n$. By Hurwitz's Theorem, G must omit T_n for every n. Since $\cup T_n$ is dense in the unit circle and $G(\mathbb{C})$ is an open connected set, this implies that either $G(\mathbb{C}) \subset \Delta$ or $G(\mathbb{C}) \subset \mathbb{C} \setminus \overline{\Delta}$. In either case, we have a contradiction to Liouville's Theorem.

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